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A TWO-LEVEL METHOD WITH BACKTRACKING FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. We consider a two-level method for resolving the nonlinearity in finite element approximations of the equilibrium Navier-Stokes equations. The method yields L^2 and H^1 optimal velocity approximations and an L^2 optimal pressure approximation. The two-level method involves solving one small, nonlinear coarse mesh system, one Oseen problem (hence, linear with positive definite symmetric part) on the fine mesh, and one linear correction problem on the coarse mesh. The algorithm we study produces an approximate solution with the optimal, asymptotic in h, accuracy for any fixed Reynolds number. We do not consider the behavior of the error for fixed h as $Re \to \infty$, i.e., for flows in transition to turbulence.

Key words. Navier–Stokes equations, finite element method, multigrid

AMS subject classifications. 65N30, 76M10

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1. Introduction.

In this report we consider a two-level method for the resolution of the nonlinear system arising from finite element discretizations of the equilibrium, incompressible Navier–Stokes equations:

$$
\left\{
\begin{array}{l}
-Re^{-1}\Delta \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \text{ in } \Omega,\\
\nabla\cdot\mathbf{u} = 0, \text{ in } \Omega,\\
\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \text{ and } \int_\Omega p\, dx = 0.
\end{array}
\right.
\tag{1.1}
$$

If (1.1) is discretized by the usual Galerkin finite element method, a large system of nonlinear algebraic equations results has the (consistently ordered) block form:

$$
\begin{bmatrix} A + N(\mathbf{u}^h) & C \\ C^T & 0 \end{bmatrix}
\begin{bmatrix} \mathbf{u}^h \\ p^h \end{bmatrix}
=
\begin{bmatrix} f^h \\ 0 \end{bmatrix}.
\tag{1.2}
$$

We are especially interested in efficient methods which can be used for all Reynolds numbers and arbitrary Ladyzhenskaya–Babuska–Brezzi (LBB) stable finite element pairs approximating the velocity $\mathbf{u}$ and pressure p, respectively, of order $k, k \geq 1$. We study herein a two-level method convergent for all Reynolds numbers and all LBB-stable velocity-pressure finite element spaces. A common choice for the solution of (1.2) is linearization by Newton's method, either on a fixed mesh [9, 11, 12] or successive meshes [18, 23, 32], and followed by solving in each Newton step the new linearized system by a multigrid method [31]. Unfortunately, the linearized problems arising from this multilevel Newton method have, at higher Reynolds number, $(1,1)$ blocks $A + N'(\mathbf{u}^n)$ which are both highly nonsymmetric and indefinite. This combination causes difficulties for linear multilevel solvers (and others as well). For example, Turek [36, section 3.1, p. 992] reports that the use of the full linearization can cause instabilities due to this combination of nonsymmetry (the $\mathbf{u}^n\cdot\nabla\mathbf{u}^{n+1}$

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