Convergence of discontinuous Galerkin methods by compactness with application to Navier–Stokes equations

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Introduction

- Discontinuous Galerkin (DG) methods were introduced in the 70's
  - hyperbolic PDE's [Reed & Hill 73, Lesaint & Raviart 74]
  - elliptic PDE's [Douglas & Dupont 76, Baker 77, Arnold 82]
- General principles and motivations
  - FE-based method using piecewise polynomials, totally discontinuous across mesh elements
  - FV-based high-order method using numerical fluxes
  - flexibility (non-matching grids, variable polynomial degree)

Introduction

- For linear PDE's, the mathematical analysis is well-understood
  - unified analysis for Poisson problem [Arnold, Brezzi, Cockburn & Marini 01]
  - unified analysis for Friedrichs' systems [AE & Guermond 06-08]

Introduction

- For linear PDE's, the mathematical analysis is well-understood
  - unified analysis for Poisson problem [Arnold, Brezzi, Cockburn & Marini 01]
  - unified analysis for Friedrichs' systems [AE & Guermond 06-08]
- For nonlinear PDE's, the situation is substantially different
  - FE-based techniques require strong regularity assumptions on the exact solution
  - the analysis of FV schemes proceeds along a different path, avoiding such assumptions [Eymard, Gallouët, Herbin et al., 00–08]
- Our goal is to extend the discrete analysis tools for FV to DG avoiding any strong regularity assumption

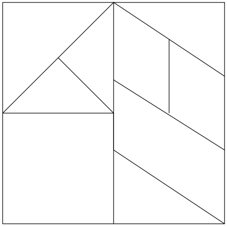
Outline

- Discrete functional analysis tools in DG spaces
- Poisson problem
- Incompressible NS

Discrete functional analysis tools in DG spaces

Admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ of bounded polyhedron $\Omega \subset \mathbb{R}^d$

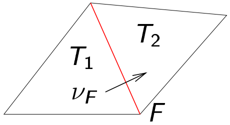
- non-conforming
- shape-regular
- $\mathrm{size}(\mathcal{T}_h) \stackrel{\mathrm{def}}{=} \max_{T\in\mathcal{T}_h} h_T$
- Example of admissible mesh



Jumps and averages

- Mesh faces: $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$
- Jumps and averages: $F = \partial T_1 \cap \partial T_2$

$$[\![\varphi]\!] \stackrel{\text{def}}{=} \varphi_{|T_1} - \varphi_{|T_2} \qquad \{\!\{\varphi\}\!\} \stackrel{\text{def}}{=} \frac{1}{2}(\varphi_{|T_1} + \varphi_{|T_2})$$



DG spaces

- $V_h^k \stackrel{\text{def}}{=} \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_{h|T} \in \mathbb{P}_k(T)\}$ norm

$$\|v_h\|_{\mathrm{DG}}^2 = \|\nabla_h v_h\|_{L^2(\Omega)^d}^2 + |v_h|_{\mathrm{J},\mathcal{F}_h,-1}^2$$

with broken gradient ∇_h and jump seminorm ($\mathcal{F} = \mathcal{F}_h$ or $\mathcal{F}_h^i$)

$$|v_h|_{\mathrm{J},\mathcal{F},\pm 1}^2 \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}} h_F^{\pm 1} \int_F |[\![v_h]\!]|^2$$

- Approximability of smooth functions For all $\varphi \in C_c^\infty(\Omega)$ and all $k \geq 1$,

$$\|\varphi - \pi_h^k \varphi\|_{\mathrm{DG}} \to 0 \quad \text{as } \mathrm{size}(\mathcal{T}_h) \to 0$$

Discrete Sobolev embeddings

- non-Hilbertian setting ($1 \le p < +\infty$)

$$\|v_h\|_{\mathrm{DG},p}^p \stackrel{\mathrm{def}}{=} \sum_{T \in \mathcal{T}_h} \int_T |\nabla v_h|_{\ell^p}^p + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F |[\![v_h]\!]|^p$$

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- Main result: For all q such that
  (i) $1 \le q \le p^* \stackrel{\mathrm{def}}{=} \frac{pd}{d-p}$ if $1 \le p < d$;
  (ii) $1 \le q < +\infty$ if $d \le p < +\infty$;

  there is $\sigma_{q,p}$ such that

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{L^q(\Omega)} \le \sigma_{p,q} \|v_h\|_{\mathrm{DG},p}$$

Discrete Sobolev embeddings

- Discrete Poincaré–Friedrichs inequality ($q = 2$, $p = 2$) [Brenner 03]
- $q = 4$, $p = 2$ [Karakashian & Jureidini 98]
- Discrete Sobolev embeddings with $p = 2$ [Lasis & Süli 03]

Discrete Sobolev embeddings

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- Discrete Sobolev embeddings with $p = 2$ [Lasis & Süli 03]
- Two key differences
  - our technique of proof is much simpler: no elliptic regularity or nonconforming FE interpolation $\Rightarrow$ general meshes can be used
  - embeddings are useful for DG spaces and not for broken Sobolev spaces

Discrete Sobolev embeddings

Principle of proof

- Inspired from [Eymard, Gallouët & Herbin 08]
- BV estimate $(\sum_{i=1}^{d} \sup\{\int_{\mathbb{R}^d} u\partial_i\varphi,\ \varphi \in C_c^\infty(\mathbb{R}^d),\ \|\varphi\|_{L^\infty(\mathbb{R}^d)} \le 1\})$

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{\mathrm{BV}} \lesssim \|v_h\|_{\mathrm{DG},1} \lesssim \|v_h\|_{\mathrm{DG},p} \quad (p \ge 1)$$

(v_h extended by zero outside Ω)

- Classical result ($1^* \stackrel{\text{def}}{=} \frac{d}{d-1}$): $\|v\|_{L^{1^*}(\mathbb{R}^d)} \le \frac{1}{2d}\|v\|_{\mathrm{BV}}$
- For $1 < p < d$, use $\|\cdot\|_{L^{1^*}(\mathbb{R}^d)}$-estimate for $|v_h|^\alpha$, Hölder's inequality and a trace inequality
- For $p \ge d$, simply use Hölder's inequality

Discrete Sobolev embeddings

- Main result for $p = 2$ and $d \in \{2,3\}$: For all q such that
  (i) $1 \leq q \leq 6$ if $d = 3$;
  (ii) $1 \leq q < +\infty$ if $d = 2$;

  there is σ_q such that

$$\forall v_h \in V_h^k, \qquad \|v_h\|_{L^q(\Omega)} \leq \sigma_q \|v_h\|_{\mathrm{DG}}$$

Discrete gradients

- Let $l \geq 0$. For all $F \in \mathcal{F}_h$, let $r_F^l : L^2(F) \rightarrow [V_h^l]^d$ s.t.

$$\forall \tau_h \in [V_h^l]^d, \qquad \int_\Omega r_F^l(\phi) \cdot \tau_h = \int_F \{\!\{\tau_h\}\!\} \cdot \nu_F \phi$$

- Support of r_F^l consists of one or two mesh elements

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- Support of r_F^l consists of one or two mesh elements
- Let $k \geq 1$, define discrete gradient $G_h^l : V_h^k \rightarrow [V_h^{\max(k-1,l)}]^d$ as

$$\forall v_h \in V_h^k, \qquad G_h^l(v_h) \stackrel{\text{def}}{=} \nabla_h v_h - \sum_{F \in \mathcal{F}_h} r_F^l([\![v_h]\!])$$

- Usual values: $l = k$ or $l = k - 1$

Discrete gradients

- Stability

$$\forall v_h \in V_h^k, \qquad \|G_h^l(v_h)\|_{L^2(\Omega)^d} \lesssim \|v_h\|_{\mathrm{DG}}$$

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- Compactness and weak convergence
  - let $\{v_h\}_{h\in\mathcal{H}}$ be a sequence in V_h^k
  - bounded in the $\|\cdot\|_{\mathrm{DG}}$-norm

  Then, there exists a subsequence of $\{v_h\}_{h\in\mathcal{H}}$ and a function $v \in H_0^1(\Omega)$ s.t. as $\mathrm{size}(\mathcal{T}_h) \to 0$,

$$v_h \to v \quad \text{strongly in } L^2(\Omega)$$

  and for all $l \geq 0$,

$$G_h^l(v_h) \rightharpoonup \nabla v \quad \text{weakly in } L^2(\Omega)^d$$

Discrete gradients

- Proof inspired from FV analysis [Eymard, Gallouët & Herbin 08]
- Uniform BV estimate on space translates

$$\|v_h(\cdot + \xi) - v_h\|_{L^1(\mathbb{R}^d)} \leq |\xi|_{\ell^1} \|v_h\|_{\mathrm{BV}} \leq C|\xi|_{\ell^1}$$

- Kolmogorov's Compactness Criterion in $L^1(\mathbb{R}^d)$
- Sobolev embedding: compactness in $L^2(\mathbb{R}^d)$
- bound on discrete gradient: $G_h^l(v_h) \rightharpoonup w$ in $L^2(\Omega)^d$

Discrete gradients

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- Uniform BV estimate on space translates

$$\|v_h(\cdot+\xi)-v_h\|_{L^1(\mathbb{R}^d)} \le |\xi|_{\ell^1}\|v_h\|_{\mathrm{BV}} \le C|\xi|_{\ell^1}$$

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- Sobolev embedding: compactness in $L^2(\mathbb{R}^d)$
- bound on discrete gradient: $G_h^l(v_h) \rightharpoonup w$ in $L^2(\Omega)^d$
- For $\varphi \in C_c^\infty(\mathbb{R}^d)^d$,

$$\int_{\mathbb{R}^d} G_h^l(v_h)\cdot\varphi = -\int_{\mathbb{R}^d} v_h(\nabla\cdot\varphi) - \int_{\mathbb{R}^d} R_h^l([\![v_h]\!])\cdot(\varphi-\pi_h^0\varphi) + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\varphi-\pi_h^0\varphi\}\!\}\cdot\nu_F[\![v_h]\!]$$

  converges to $-\int_{\mathbb{R}^d} v(\nabla\cdot\varphi)$
- $\nabla v = w$, $v\in H^1(\mathbb{R}^d)$, and $v\equiv 0$ outside $\Omega \Rightarrow v\in H_0^1(\Omega)$.

Poisson problem

- A basic formulation
- Convergence analysis
- Nonsymmetric variants

A basic formulation

- Let $f \in L^r(\Omega)$ with $r \geq \frac{6}{5}$ if $d = 3$ and $r > 1$ if $d = 2$
- $u \in H_0^1(\Omega)$ s.t. for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv$$

A basic formulation

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- $u \in H^1_0(\Omega)$ s.t. for all $v \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv$$

- DG bilinear form (disc. grad. with $l = k$ or $k - 1$)

$$a_h(v_h, w_h) \stackrel{\text{def}}{=} \int_\Omega G_h(v_h) \cdot G_h(w_h) + j_h(v_h, w_h)$$

- Stabilization

$$j_h(v_h, w_h) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \eta \int_\Omega r_F([\![v_h]\!]) \cdot r_F([\![w_h]\!]) - \int_\Omega R_h([\![v_h]\!]) \cdot R_h([\![w_h]\!])$$

A basic formulation

- Stabilization parameter $\eta > N_\partial$ (max. number of faces per mesh element)
- Stability result: For all $v_h \in V_h^k$,

$$\|G_h(v_h)\|^2_{L^2(\Omega)^d} + (\eta - N_\partial) \sum_{F \in \mathcal{F}_h} \|r_F([\![v_h]\!])\|^2_{L^2(\Omega)^d} \leq a_h(v_h, v_h)$$

- Coercivity: $\exists \alpha > 0$ s.t. for all $v_h \in V_h^k$,

$$\alpha \|v_h\|^2_{\mathrm{DG}} \leq a_h(v_h, v_h)$$

Variants on stabilization

- Expanding the lifting operators yields

$$a_h(v_h, w_h) = \int_\Omega \nabla_h v_h \cdot \nabla_h w_h + \sum_{F \in \mathcal{F}_h} \eta \int_\Omega r_F([\![v_h]\!]) \cdot r_F([\![w_h]\!])$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \left(\nu_F \cdot \{\!\{\nabla_h v_h\}\!\} [\![w_h]\!] + \nu_F \cdot \{\!\{\nabla_h w_h\}\!\} [\![v_h]\!] \right)$$

This is the IP-method of Bassi, Rebay et al. 97

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- SIPG method [Arnold 82] and LDG method [Cockburn & Shu 98]

$$j_h^{\mathrm{SIPG}}(v_h, w_h) \stackrel{\mathrm{def}}{=} \sum_{F \in \mathcal{F}_h} \eta \frac{1}{h_F} \int_F [\![v_h]\!][\![w_h]\!] - \int_\Omega R_h([\![v_h]\!]) \cdot R_h([\![w_h]\!])$$

$$j_h^{\mathrm{LDG}}(v_h, w_h) \stackrel{\mathrm{def}}{=} \sum_{F \in \mathcal{F}_h} \eta \frac{1}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

Convergence result

Let $\{u_h\}_{h\in\mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete Poisson problem on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as $\mathrm{size}(\mathcal{T}_h) \to 0$,

$$
\begin{aligned}
u_h &\to u && \text{in } L^2(\Omega)\\
G_h(u_h) &\to \nabla u && \text{in } L^2(\Omega)^d\\
\nabla_h u_h &\to \nabla u && \text{in } L^2(\Omega)^d\\
|u_h|_{\mathrm{J},\mathcal{F}_h,-1} &\to 0
\end{aligned}
$$

where $u \in H^1_0(\Omega)$ is the exact solution

Sketch of proof

- A priori estimate:

$$\alpha \|u_h\|_{\mathrm{DG}}^2 \le a(u_h, u_h) = \int_\Omega f u_h \le \|f\|_{L^r(\Omega)} \|u_h\|_{L^{r'}(\Omega)}$$

and Sobolev embedding yields

$$\|u_h\|_{\mathrm{DG}} \le C$$

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  and Sobolev embedding yields

$$\|u_h\|_{\mathrm{DG}} \leq C$$

- Compactness: there exists a subsequence of $\{u_h\}_{h\in\mathcal{H}}$ and $u \in H_0^1(\Omega)$ s.t. as $\mathrm{size}(\mathcal{T}_h) \to 0$,

$$u_h \to u \qquad \text{strongly in } L^2(\Omega)$$
$$G_h(u_h) \rightharpoonup \nabla u \qquad \text{weakly in } L^2(\Omega)^d$$

Sketch of proof

- Identification of the limit: For all $\varphi \in C_c^\infty(\Omega)$,

$$a_h(u_h, \pi_h\varphi) \to \int_\Omega \nabla u \cdot \nabla \varphi$$

so that

$$\int_\Omega f\varphi \leftarrow \int_\Omega f\pi_h\varphi = a_h(u_h, \pi_h\varphi) \to \int_\Omega \nabla u \cdot \nabla \varphi$$

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- By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, u solves the Poisson problem
- By uniqueness of the solution, the whole sequence converges

Sketch of proof

- Owing to weak convergence

$$\liminf \|G_h(u_h)\|^2_{L^2(\Omega)^d} \geq \|\nabla u\|^2_{L^2(\Omega)^d}$$

- Owing to stability

$$\|G_h(u_h)\|^2_{L^2(\Omega)^d} \leq a_h(u_h, u_h) = \int_\Omega f u_h$$

  so that

$$\limsup \|G_h(u_h)\|^2_{L^2(\Omega)^d} \leq \limsup \int_\Omega f u_h = \int_\Omega f u = \|\nabla u\|^2_{L^2(\Omega)^d}$$

- Hence, $\|G_h(u_h)\|_{L^2(\Omega)^d} \to \|\nabla u\|_{L^2(\Omega)^d}$ so that $G_h(u_h)$ strongly converges to ∇u in $L^2(\Omega)^d$

Sketch of proof

- Owing to stability

$$(\eta - N_\partial) \sum_{F \in \mathcal{F}_h} \|r_F([\![u_h]\!])\|^2_{L^2(\Omega)^d} \leq a_h(u_h, u_h) - \|G_h(u_h)\|^2_{L^2(\Omega)^d}$$

- Hence, $|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0$

Sketch of proof

- Owing to stability

$$(\eta - N_\partial) \sum_{F \in \mathcal{F}_h} \|r_F([\![u_h]\!])\|^2_{L^2(\Omega)^d} \leq a_h(u_h, u_h) - \|G_h(u_h)\|^2_{L^2(\Omega)^d}$$

- Hence, $|u_h|_{\mathrm{J},\mathcal{F}_h,-1} \to 0$

Remark. If the exact solution is smooth, the usual optimal a priori error estimates are recovered

$$\|u - u_h\|_{\mathrm{DG}} \leq C(u)h^k$$

Nonsymmetric variants

- Nonsymmetric DG bilinear form

$$a_h(v_h, w_h) = \int_\Omega \widehat{G}_h(v_h)\cdot G_h(w_h) + j'_h(v_h, w_h)$$

- Design conditions
  - $\widehat{G}_h$ strongly consistent for smooth functions
  - G_h weakly consistent for discrete functions
  - both gradients controlled by $\|\cdot\|_{\mathrm{DG}}$-norm
  - j'_h symmetric, nonnegative, controlled by jump seminorm and ensuring coercivity of a_h

Nonsymmetric variants

- General convergence result can be proven as before
- Examples of nonsymmetric methods

$$G_h(v_h) = \nabla_h v_h + R_h([\![v_h]\!]) \qquad \text{(NIPG)}$$
$$G_h(v_h) = \nabla_h v_h \qquad \text{(IIPG)}$$

Incompressible Navier–Stokes

- Pressure-velocity coupling (Stokes system)
- Convective trilinear form for NS
- Convergence result

Stokes system

- Let $f \in L^r(\Omega)^d$ with $r \geq \frac{6}{5}$ if $d = 3$ and $r > 1$ if $d = 2$
- Let $\nu > 0$
- $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ s.t. for all $(v, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\nu \int_\Omega \nabla u \cdot \nabla v - \int_\Omega p \nabla \cdot v + \int_\Omega q \nabla \cdot u = \int_\Omega f \cdot v$$

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- $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ s.t. for all $(v, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\nu \int_\Omega \nabla u \cdot \nabla v - \int_\Omega p \nabla \cdot v + \int_\Omega q \nabla \cdot u = \int_\Omega f \cdot v$$

- Equal-order polynomial spaces for velocity and pressure

$$U_h \stackrel{\text{def}}{=} [V_h^k]^d \qquad P_h \stackrel{\text{def}}{=} V_h^k \qquad X_h \stackrel{\text{def}}{=} U_h \times P_h$$

- Pressure stabilization $s_h(q_h, r_h) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h^i} h_F \int_F [\![q_h]\!][\![r_h]\!]$

Pressure–velocity coupling

- Discrete divergence operator

$$\forall v_h \in U_h, \qquad D_h^l(v_h) = G_h^l(v_{h,j}) \cdot e_j$$

- Pressure–velocity bilinear form

$$b_h(v_h, q_h) \stackrel{\text{def}}{=} -\int_\Omega q_h D_h^k(v_h)$$

- $(u_h, p_h) \in X_h$ s.t. $l_h((u_h, p_h), (v_h, q_h)) = \int_\Omega f \cdot v_h, \ \forall (v_h, q_h) \in X_h$ where

$$l_h((u_h, p_h), (v_h, q_h)) \stackrel{\text{def}}{=} \nu a_h(u_{h,i}, u_{h,i}) + b_h(v_h, p_h) - b_h(u_h, q_h) + s_h(p_h, q_h)$$

Convergence result

Let $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete Stokes problems on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as $\mathrm{size}(\mathcal{T}_h) \to 0$,

$$
\begin{aligned}
u_h &\to u && \text{in } L^2(\Omega)^d \\
\nabla_h u_h &\to \nabla u && \text{in } L^2(\Omega)^{d,d} \\
|u_h|_{\mathrm{J},\mathcal{F}_h,-1} &\to 0 && \\
p_h &\to p && \text{in } L^2(\Omega) \\
|p_h|_{\mathrm{J},\mathcal{F}_h^i,1} &\to 0 &&
\end{aligned}
$$

where $(u, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ is the exact Stokes solution

Sketch of proof

- Coercivity on velocity and discrete inf-sup condition on pressure
- A priori estimate + compactness: $u_h \to u$ strongly in $L^2(\Omega)^d$, $G_h(u_{h,i}) \rightharpoonup \nabla u_i$ weakly in $L^2(\Omega)^d$ and $p_h \rightharpoonup p$ weakly in $L^2(\Omega)$
- Identification of the limit and convergence of the whole sequence
- Strong convergence of velocity gradient and jumps (as before)
- Strong convergence of the pressure using Nečas velocity

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Remark. If the exact solution is smooth, the usual optimal a priori error estimates are recovered [Cockburn, Kanschat, Schötzau & Schwab 02, AE & Guermond 08]

$$\|(u - u_h, p - p_h)\|_{\mathrm{S}} \leq C(u)h^k$$

Incompressible NS system

- Let $f \in L^r(\Omega)^d$ with $r \geq \frac{6}{5}$ if $d = 3$ and $r > 1$ if $d = 2$
- Let $\nu > 0$
- $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ s.t. for all $(v, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\nu \int_\Omega \nabla u \cdot \nabla v + \int_\Omega v \cdot (\nabla \cdot F(u, p)) + \int_\Omega q \nabla \cdot u = \int_\Omega f \cdot v$$

  with incomp. Euler flux $F(u, p) = u \otimes u + pI$
- Existence of such a weak solution holds for $d \in \{2, 3\}$
- Uniqueness under small data assumption

Incompressible NS system

- For all $u \in H_0^1(\Omega)^d$,

$$\int_\Omega u\cdot\nabla\cdot(u \otimes u) = \int_\Omega u\cdot(\tfrac{1}{2}(\nabla\cdot u)u) = -\int_\Omega u\cdot\nabla(\tfrac{1}{2}|u|^2)$$

- Temam's device for stability: add source term $-\int_\Omega \frac{1}{2}(\nabla\cdot u)u$
  - non-conservative form
  - source term vanishes at the limit for solenoidal velocity
- Modified Euler flux $\Phi(u,\overline{p}) = u \otimes u + \frac{1}{2}|u|^2 I + \overline{p}I$ with $\overline{p} = p - \frac{1}{2}|u|^2$
  - conservative form
  - hinted to in [Cockburn, Kanschat & Schötzau 05]

Discrete NS system

- DG methods for incompressible NS
  - piecewise solenoidal velocity fields [Karakashian & Jureidini 98]
  - nonconservative method based on Temam's device [Girault, Rivière & Wheeler 04]
  - conservative LDG method [Cockburn, Kanschat & Schötzau 04] using BDM projection
- FV methods for incompressible NS
  - nonconservative form [Eymard, Herbin & Latché 07]
  - conservative form [Chénier, Eymard & Herbin 08]

Discrete NS system

- $(u_h, p_h) \in X_h$ s.t. $\forall (v_h, q_h) \in X_h$,

$$l_h((u_h, p_h), (v_h, q_h)) + t_h(u_h, u_h, v_h) = \int_\Omega f \cdot v_h$$

  with Stokes bilinear form l_h and discrete trilinear form t_h

- Design conditions on t_h
  - Stability: $t_h(v_h, v_h, v_h) = 0, \forall v_h \in U_h$
  - Continuity on discrete space
  - Weak continuity: $t_h(u_h, u_h, \pi_h \varphi) \to t(u, u, \varphi)$

- Existence of discrete solution using topological degree argument (no small data assumption!)

Convergence result for NS

Let $\{(u_h, p_h)\}_{h\in\mathcal{H}}$ be a sequence of approximate solutions generated by solving the discrete NS problems on the admissible meshes $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$. Then, as $\mathrm{size}(\mathcal{T}_h) \to 0$, up to a subsequence

$$
\begin{aligned}
u_h &\to u && \text{in } L^2(\Omega)^d \\
\nabla_h u_h &\to \nabla u && \text{in } L^2(\Omega)^{d,d} \\
|u_h|_{\mathrm{J},\mathcal{F}_h,-1} &\to 0 && \\
p_h &\to p && \text{in } L^2(\Omega) \\
|p_h|_{\mathrm{J},\mathcal{F}_h^i,1} &\to 0 &&
\end{aligned}
$$

where $(u, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ is an exact solution

Examples of DG trilinear forms

- Non-conservative, based on Temam's device

$$t_h(w,u,v) = \int_\Omega (w\cdot\nabla_h u)\cdot v - \sum_{F\in\mathcal{F}_h^i} \int_F \{\!\{w\}\!\}\cdot\nu_F [\![u]\!]\cdot\{\!\{v\}\!\}$$

$$+ \int_\Omega \tfrac{1}{2}\nabla_h\cdot w(u\cdot v) - \sum_{F\in\mathcal{F}_h} \int_F [\![w]\!]\cdot\nu_F \tfrac{1}{2}\{\!\{u\cdot v\}\!\}$$

- Conservative, based on Euler flux modification

$$t_h(w,u,v) = -\int_\Omega (w\otimes u):\nabla_h v + \sum_{F\in\mathcal{F}_h^i} \int_F \nu_F\cdot\{\!\{u\}\!\}\{\!\{w\}\!\}\cdot[\![v]\!]$$

$$+ \int_\Omega \tfrac{1}{2} v\cdot\nabla_h(u\cdot w) - \sum_{F\in\mathcal{F}_h^i} \int_F \nu_F\cdot\{\!\{v\}\!\}\tfrac{1}{2}[\![u\cdot w]\!]$$

Concluding remarks

- Uniqueness of discrete solution under small data assumption
- Upwinding of convective term
- Optimal a priori error analysis under strong regularity assumptions
- Confirmed by numerical tests on standard benchmark problems with moderate Reynolds (≤ 100)
- For higher Reynolds numbers, the artifical compressibility method of [Bassi, Di Pietro & Rebay 07], yet to be analyzed mathematically, yields better CV of nonlinear solver