On the choice of stabilization parameters in the local projection method

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Local projection stabilization I

Class of problems

- dominated convection in Convection-diffusion equations
- equal order interpolations for Stokes
- both effects in Oseen and Navier-Stokes equations
- dominated convection for inf-sup stable elements

References

- Guermond 99, Codina 00, Becker/Braack 01, 04, Braack/Burman 06, Matthies/Skrzypacz/To 07, Lube/Rapin/Löwe 07, Ganesan/Matthies/To 08, Braack 08, Knobloch/To 08. . .

Local projection stabilization II

general Idee

- add a stabilization term controlling fluctuations of gradients (gradients of fluctuations) of the quantity of interest

$$+\sum_{K\in\mathcal{T}_h}\tau_K\left(\kappa_h(b\cdot\nabla u),\kappa_h(b\cdot\nabla v)\right)_K$$

$$+\sum_{K\in\mathcal{T}_h}\alpha_K\left(\kappa_h(\nabla p),\kappa_h(\nabla q)\right)_K$$

$$+\sum_{K\in\mathcal{T}_h}\tau_K\left(\kappa_h(b\cdot\nabla u),\kappa_h(b\cdot\nabla v)\right)_K+\sum_{K\in\mathcal{T}_h}\alpha_K\left(\kappa_h(\nabla p),\kappa_h(\nabla q)\right)_K$$
$$+\sum_{K\in\mathcal{T}_h}\mu_K\left(\kappa_h(\operatorname{div} u),\kappa_h(\operatorname{div} v)\right)_K$$

Local projection stabilization III

$\kappa_h = \mathrm{id} - \pi_h$, $\pi_h : L^2(\Omega) \to D_h$ denotes L^2 projection
(V_h, D_h) pairs of approximation and projection spaces

- D_h rich enough to guarentee a certain order of consistency
- D_h small enough w.r.t. V_h to guarantee $j_h u - u \perp D_h$

two variants

- one-level approach (V_h^+, D_h)
- two-level approach (V_h, D_{2h})

How to choose τ_K, α_K, μ_K?

One level LPS in one space dimension

Model problem

$$-\varepsilon u'' + bu' + cu = f \quad \text{in } (0,1), \qquad u(0) = u(1) = 0$$

$$(V_h^+, D_h) = (P_r^+, P_{r-1}^{\text{disc}}) = (P_{r+1}, P_{r-1}^{\text{disc}})$$

Stabilized method

Find $u_h^+ \in V_h^+$ such that for all $v_h^+ \in V_h^+$

$$\varepsilon((u_h^+)', (v_h^+)') + (b(u_h^+)' + cu_h^+, v_h^+) + \sum_{K \in \mathcal{T}_h} \tau_K(\kappa_h(b(u_h^+)'), \kappa_h(b(v_h^+)')) = (f, v_h^+)$$

Elimination of enrichments I

Set $K = (x_i, x_{i+1})$, $h_K = x_{i+1} - x_i$ and split the space

$$V_h^+ = V_h \oplus B_h, \qquad B_h = \text{span} \bigoplus_{K \in \mathcal{T}_h} \varphi_{r,K}$$

where the bubble space is spanned by

$$\varphi_{r,K}(x) = \begin{cases} L_{r+1}\left(\frac{2x - x_i - x_{i+1}}{h_K}\right) - L_{r-1}\left(\frac{2x - x_i - x_{i+1}}{h_K}\right) & \text{for} \quad x \in K \\ 0 & \text{for} \quad x \notin K \end{cases}$$

Important properties

$$(v_h', \varphi_{r,K}') = 0 \quad \forall v_h \in V_h, \ \forall K \in \mathcal{T}_h$$

$$\pi_h v_h' = v_h' \quad \forall v_h \in V_h \quad \Leftrightarrow \quad \kappa_h v_h' = 0 \quad \forall v_h \in V_h$$

Elimination of enrichments II

Case $b = const$, $c = 0$, and f piecewise P_{r-1}, using

$$(b\varphi'_{r,K}, \varphi_{r,K}) = 0, \quad \pi_h\varphi'_{r,K} = 0, \quad \varphi_{r,K} = \psi_K^{(r-1)}, \quad \Psi_K^{(j)} = 0 \text{ on } \partial K$$

for $j = 0, 1, \ldots, r-1$ we get

$$\varepsilon(u'_h, v'_h) + (bu'_h, v_h) + \sum_{K\in\mathcal{T}_h} u_K(b\varphi'_{r,K}, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

$$u_K\left\{\varepsilon(\varphi'_{r,K}, \varphi'_{r,K}) + \tau_K b^2(\varphi'_{r,K}, \varphi'_{r,K})\right\} = (f - bu'_h, \varphi_{r,K}) \quad \forall K \in \mathcal{T}_h$$

$$\boxed{u_K = (f - bu'_h)^{(r-1)}|_K \frac{(-1)^{r-1}(1, \psi_K)}{(\varepsilon + \tau_K b^2)|\varphi_{r,K}|^2_{1,K}}}$$

Differentiated residual method DRM – SUPG (r=1)

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\varepsilon(u_h', v_h') + (bu_h' + cu_h, v_h) + \sum_{K \in \mathcal{T}_h} \gamma_K \left((bu_h' + cu_h)^{(r-1)}, (bv_h')^{(r-1)}\right)_K$$
$$= (f, v_h) + \sum_{K \in \mathcal{T}_h} \gamma_K \left(f^{(r-1)}, (bv_h')^{(r-1)}\right)_K$$

$$\boxed{\gamma_K = \frac{(1, \psi_K)^2}{(\varepsilon + \tau_K b^2) h_K |\varphi_{r,K}|_{1,K}^2}}$$

References

- Hughes/Sangalli 05/07, To 06

Relationship LPS and DRM

Theorem

Assume $b = const$, $c = 0$, and f piecewise P_{r-1}. Eliminating the enrichment in the (P_r^+, P_{r-1}^{disc})-LPS gives the P_r-DRM with the correct scaling in both the convection dominated and diffusion dominated limit.

How to choose γ_K?

Recursion formula for P_{r+1}-DRM I

Assume $b = const$, $c = 0$, and f piecewise P_{r-1}. Set

$$V_r = \{v_h \in H_0^1(0,1) : v_h| \in P_r(K),\ K \in \mathcal{T}_h\}, \quad r \geq 1$$

und use the splitting

$$V_{r+1} = V_r \oplus \operatorname{span} \bigoplus_{K \in \mathcal{T}_h} \varphi_{r,K}\,.$$

$$\varepsilon(u_r', v_r') + (bu_r', v_r) + \sum_{K \in \mathcal{T}_h} u_K(b\varphi_{r,K}', v_r) = (f, v_r) \qquad \forall v_r \in V_r$$

$$u_K \left\{ \varepsilon|\varphi_{r,K}|_1^2 + \gamma_{r+1} b^2 |\varphi_{r,K}|_{r+1}^2 \right\} = (f - bu_r', \varphi_{r,K}) \quad \forall K \in \mathcal{T}_h$$

Recursion formula for P_{r+1}-DRM II

Integrating by parts

$$(f - bu_r', \varphi_{r,K}) = (f - bu_r', \psi_K^{(r-1)}) = (-1)^{r-1}(f - bu_r')^{(r-1)}|_K\,(1, \psi_K)$$

$$(b\varphi_{r,K}', v_r) = -(bv_r', \psi_K^{(r-1)}) = (-1)^r (bv_r')^{(r-1)}|_K\,(1, \psi_K)$$

P_r-DRM

$$\varepsilon(u_r', v_r') + (bu_r', v_r) + \sum_{K \in \mathcal{T}_h} \gamma_r((bu_r')^{(r-1)}, (bv_r')^{(r-1)})_K = (f, v_r) + \sum_{K \in \mathcal{T}_h} \gamma_r(f^{(r-1)}, (bv_r')^{(r-1)})_K$$

Recursion formula for P_{r+1}-DRM III

Theorem

Assume $b = const$, $c = 0$, and f piecewise P_{r-1}. Eliminating the highest order mode in the P_{r+1}-DRM results in the P_r-DRM. The P_1-DRM is equal to the SUPG.

$$\gamma_r = \frac{(1, \psi_K)^2}{\left\{\varepsilon|\varphi_{r,K}|_1^2 + \gamma_{r+1} b^2 |\varphi_{r,K}|_{r+1}^2\right\} h_K}$$

DRM stabilization parameters, $r \geq 1$

Optimal SUPG parameter (nodal exact solution in the constant coefficient case) leads to

$$\gamma_r = \frac{h_K^{2r-1}}{\alpha_r\, b}\Phi_r(q_K), \quad q_K = \frac{b_K h_K}{2\varepsilon}, \qquad \alpha_r = \frac{2[(2r-1)!]^2}{[(r-1)!]^2}$$

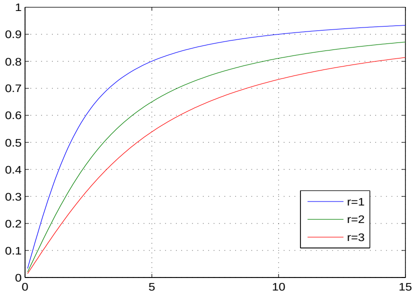
$$\Phi_{r+1}(q) = \frac{1}{\Phi_r(q)} - \frac{1}{q}, \quad \Phi_1(q) = \coth q - \frac{1}{q}.$$

Observations

- Theory $\tau_K \sim \tau_0 h_K$ but τ_0 ???
- P_1^+ discretization
  - $\tau_0 \gg 1$ oscillations
  - $\tau_0 \to 0$ smearing
- P_2^+ discretization
  - $\tau_0 \gg 1$ smearing
  - $\tau_0 \to 0$ oscillations

Weighting functions

$$\Phi_1(q) = \coth q - \frac{1}{q}, \qquad \Phi_{r+1}(q) = \frac{1}{\Phi_r} - \frac{2r+1}{q}, \quad r = 1, 2, \ldots$$



$$\lim_{q \to +0} \Phi_r(q) = 0, \qquad \lim_{q \to \infty} \Phi_r(q) = 1, \quad r = 1, 2, \ldots$$

Numerical tests

Example 1

Exponential boundary layer at $x = 0$

$$-\varepsilon u'' - (1+x^2)u' + \left(x - \frac{1}{2}\right)^2 u = 4(3x^2 - 3x + 1)(1+x)^2$$

$$u(0) = -1, \qquad u(1) = 0$$

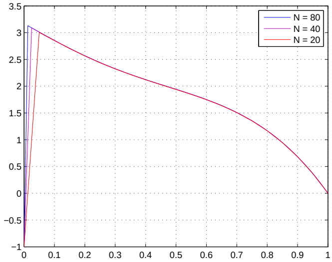
- $\varepsilon = 10^{-7}$, P_r-SUPG $r = 1, 2$, P_r-DRM $r = 1, 2, 3$
- b_K piecewise constant approximation of $b = -(1+x^2)$
- $\tau_r = \Phi_r(q_K) h_K^{2r-1}/(\alpha_r b_K), \quad q_K = b_K h_K/(2\varepsilon)$
  $\alpha_1 = 2$, $\alpha_2 = 72$, $\alpha_3 = 7200$

SUPG on equidistant mesh

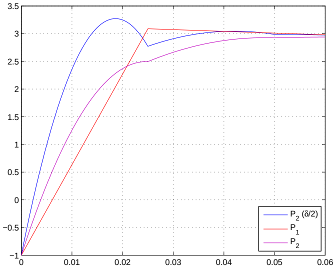
SUPG parameter

$$\delta_K = \frac{h_K}{2b_K}\Phi_1(q_K), \qquad q_K = \frac{b_K h_K}{2\varepsilon}$$

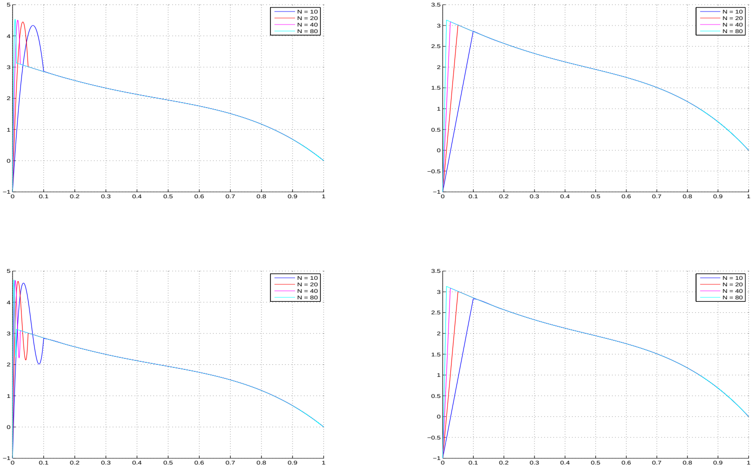
piecewise linears



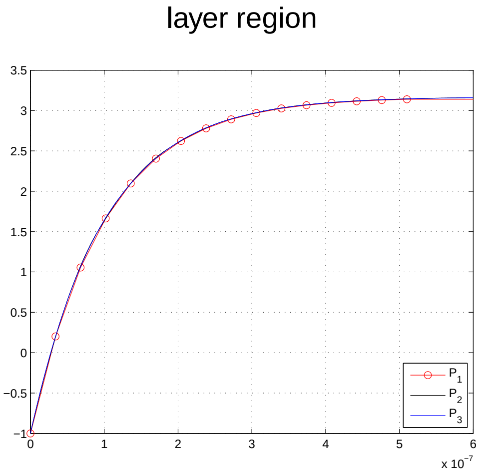
boundary layer region



P_r DMR $r = 2, 3$ on equidistant mesh



P_r DMR $r = 1, 2, 3$ on Shishkin mesh



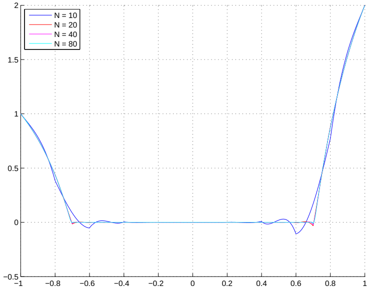
Numerical tests
Example 2

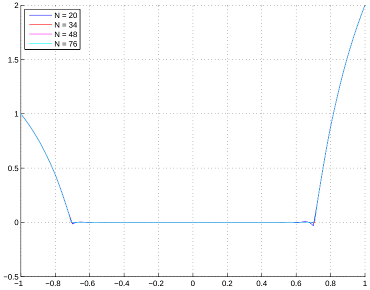
Interior layer in the first derivative at $\pm 1/\sqrt{2}$

$$-\varepsilon u'' - \left(x^3 - \frac{x}{2}\right) u' + u = 0, \qquad u(-1) = 1, \qquad u(1) = 2$$

- $\varepsilon = 10^{-7}$, P_r-DRM, $r = 1, 2, 3$
- b_K piecewise constant approximation of $b = -(x^3 - x/2)$
- $\tau_r = \Phi_r(q_K) h_K^{2r-1} / (\alpha_r b_K), \quad q_K = b_K h_K / (2\varepsilon)$
  $\alpha_1 = 2$, $\alpha_2 = 72$, $\alpha_3 = 7200$

P_3 DMR on equidistant meshes





Numerical tests

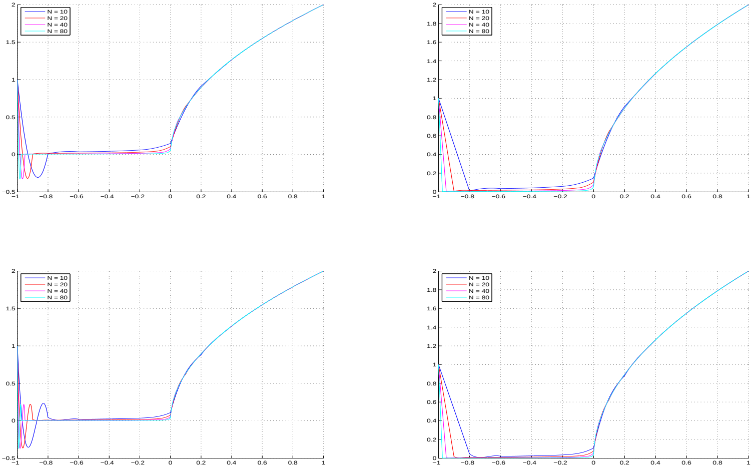
Example 3

Exponential boundary layer at $x = 0$ and interior layer in the first derivative at $x = 0$

$$-\varepsilon u'' - |x|u' + \frac{1}{2}u = 0, \qquad u(-1) = 1, \qquad u(1) = 2$$

- $\varepsilon = 10^{-7}$, P_r-DRM, $r = 1, 2, 3$
- b_K piecewise constant approximation of $b = -(x^3 - x/2)$
- $\tau_r = \Phi_r(q_K)h_K^{2r-1}/(\alpha_r b_K), \quad q_K = b_K h_K/(2\varepsilon)$
  $\alpha_1 = 2$, $\alpha_2 = 72$, $\alpha_3 = 7200$

P_r DMR $r = 2, 3$ on equidistant mesh



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