Local projection methods for convection–diffusion equations

Petr Knobloch
Charles University, Prague

joint work with

Gert Lube (Göttingen) and **Lutz Tobiska** (Magdeburg)

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Local projection stabilizations

Becker, Braack (2001) Stokes

Becker, Braack (2004) transport, Navier–Stokes

Braack, Burman (2006) Oseen

Braack, Richter (2006) Stokes

Braack, Richter (2006) Navier–Stokes

Braack, Richter (2006, 2007) reactive flows

Becker, Vexler (2007) conv.–diff.–react., optimal control

Lube, Rapin, Löwe (2007) Oseen

Ganesan, Tobiska (2007) conv.–diff.–react., Stokes, Oseen

Matthies, Skrzypacz, Tobiska (2007) Oseen, enrichment

Matthies, Skrzypacz, Tobiska (2007) conv.–diff.–react.

Knobloch, Lube (2008) conv.–diff.–react.

Knobloch, Tobiska (2008) conv.–diff.–react.

Steady convection–diffusion–reaction equation

$$-\varepsilon\,\Delta u + \mathbf{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega,$$

$$u = u_b \quad \text{on } \Gamma^D, \qquad\qquad \varepsilon\,\frac{\partial u}{\partial \mathbf{n}} = g \quad \text{on } \Gamma^N$$

$\Omega \subset \mathbb{R}^d$, $d = 2,3$ ... bounded domain with a polyhedral Lipschitz–continuous boundary $\partial\Omega$

$\Gamma^D, \Gamma^N \subset \partial\Omega$ ... relatively open, disjoint

$\overline{\Gamma^D \cup \Gamma^N} = \partial\Omega, \quad \mathrm{meas}_{d-1}(\Gamma^D) > 0$

$\mathbf{n}$ ... outer unit normal vector to $\partial\Omega$

$\varepsilon > 0$ constant, $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$,

$u_b \in H^{1/2}(\Gamma^D)$, $g \in H^{-1/2}(\Gamma^N)$, $\sigma := c - \frac{1}{2}\,\mathrm{div}\,\mathbf{b} \ge \sigma_0 > 0$

$$\{\mathbf{x} \in \Omega;\ (\mathbf{b}\cdot\mathbf{n})(\mathbf{x}) < 0\} \subset \Gamma^D$$

Steady convection–diffusion–reaction equation

$$-\varepsilon\,\Delta u + \mathbf{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega,$$

$$u = u_b \quad \text{on } \Gamma^D, \qquad\qquad \varepsilon\,\frac{\partial u}{\partial \mathbf{n}} = g \quad \text{on } \Gamma^N$$

Weak formulation

Find $u \in H^1(\Omega)$ such that $u = u_b$ on Γ^D and

$$a(u,v) = (f,v) + \langle g, v\rangle_{\Gamma^N} \qquad \forall\, v \in V\,,$$

where $\quad V := \{v \in H^1(\Omega)\,;\ v = 0 \ \text{ on } \Gamma^D\}\,,$

$$a(u,v) = \varepsilon\,(\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + (c\,u, v)$$

$\exists$ unique weak solution

Discrete problem

$\mathscr{T}_h$ ... triangulation of Ω consisting of closed shape–regular cells T (simplices, quadrilaterals or hexahedra) with usual compatibility properties

$\mathscr{M}_h$ ... coarse triangulation constructed by coarsening the triangulation $\mathscr{T}_h$ such that each macro–element $M \in \mathscr{M}_h$ is the union of one or more neighboring cells $T \in \mathscr{T}_h$. Elements of $\mathscr{M}_h$ are non–overlapping and shape–regular, $h_M \leq C\, h_T \;\forall\, T \in \mathscr{T}_h, M \in \mathscr{M}_h$ with $T \subset M$

$W_h \subset H^1(\Omega)$ ... FE space on $\mathscr{T}_h$

$V_h := W_h \cap V$

D_h ... discontinuous FE space on $\mathscr{M}_h$

π_h ... orthogonal L^2 projection of $L^2(\Omega)$ onto D_h

$\kappa_h := id - \pi_h$ ... fluctuation operator

Discrete problem

Find $u_h \in W_h$ such that $u_h - \widetilde{u}_{bh} \in V_h$ and $(\widetilde{u}_{bh}|_{\Gamma^D} \sim u_b)$

$$a_h(u_h, v_h) = (f, v_h) + \langle g, v_h \rangle_{\Gamma^N} \qquad \forall\, v_h \in V_h\,,$$

where $$a_h(u,v) = a(u,v) + s_h(u,v),$$

$$s_h(u,v) = \sum_{M \in \mathscr{M}_h} \tau_M\, s_M(u,v)\,, \qquad \tau_M \geq 0$$

and

$$s_M(u,v) = (\kappa_h(\mathbf{b}\cdot\nabla u), \kappa_h(\mathbf{b}\cdot\nabla v))_M \quad \ldots \quad \text{SD–based LPS}$$

or

$$s_M(u,v) = (\kappa_h \nabla u, \kappa_h \nabla v)_M \quad \ldots \quad \text{gradient–based LPS}$$

Two variants of LPS

One–level approach: $\mathscr{M}_h = \mathscr{T}_h$

spaces W_h/D_h :

$$P^{bub}_{k,\mathscr{T}_h} \cap H^1(\Omega)/P_{k-1,\mathscr{T}_h}$$
$$Q^{bub}_{k,\mathscr{T}_h} \cap H^1(\Omega)/P_{k-1,\mathscr{T}_h}$$
$$Q^{bub}_{k,\mathscr{T}_h} \cap H^1(\Omega)/Q_{k-1,\mathscr{T}_h}$$

Two–level approach: $\mathscr{T}_h$ is obtained by a refinement of $\mathscr{M}_h$

spaces W_h/D_h :

$$P_{k,\mathscr{T}_h} \cap H^1(\Omega)/P_{k-1,\mathscr{M}_h}$$
$$Q_{k,\mathscr{T}_h} \cap H^1(\Omega)/P_{k-1,\mathscr{M}_h}$$
$$Q_{k,\mathscr{T}_h} \cap H^1(\Omega)/Q_{k-1,\mathscr{M}_h}$$

can be viewed as one–level approach for simplicial meshes

Inf–sup condition

$$\exists\, \beta > 0 : \qquad \inf_{q_h \in D_h(M)} \; \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M}\, \|q_h\|_{0,M}} \geq \beta \qquad \forall\, M \in \mathscr{M}_h$$

where $Y_h(M) = H_0^1(M) \cap \{v_h|_M \,;\, v_h \in W_h\}$

Necessary condition: $\dim Y_h(M) \geq \dim D_h(M)$

Sufficient condition: $b_M \cdot D_h(M) \subset Y_h(M) \qquad \forall\, M \in \mathscr{M}_h$

and all macro–elements in $\mathscr{M}_h$ are affine equivalent to a reference element $\hat{T}$

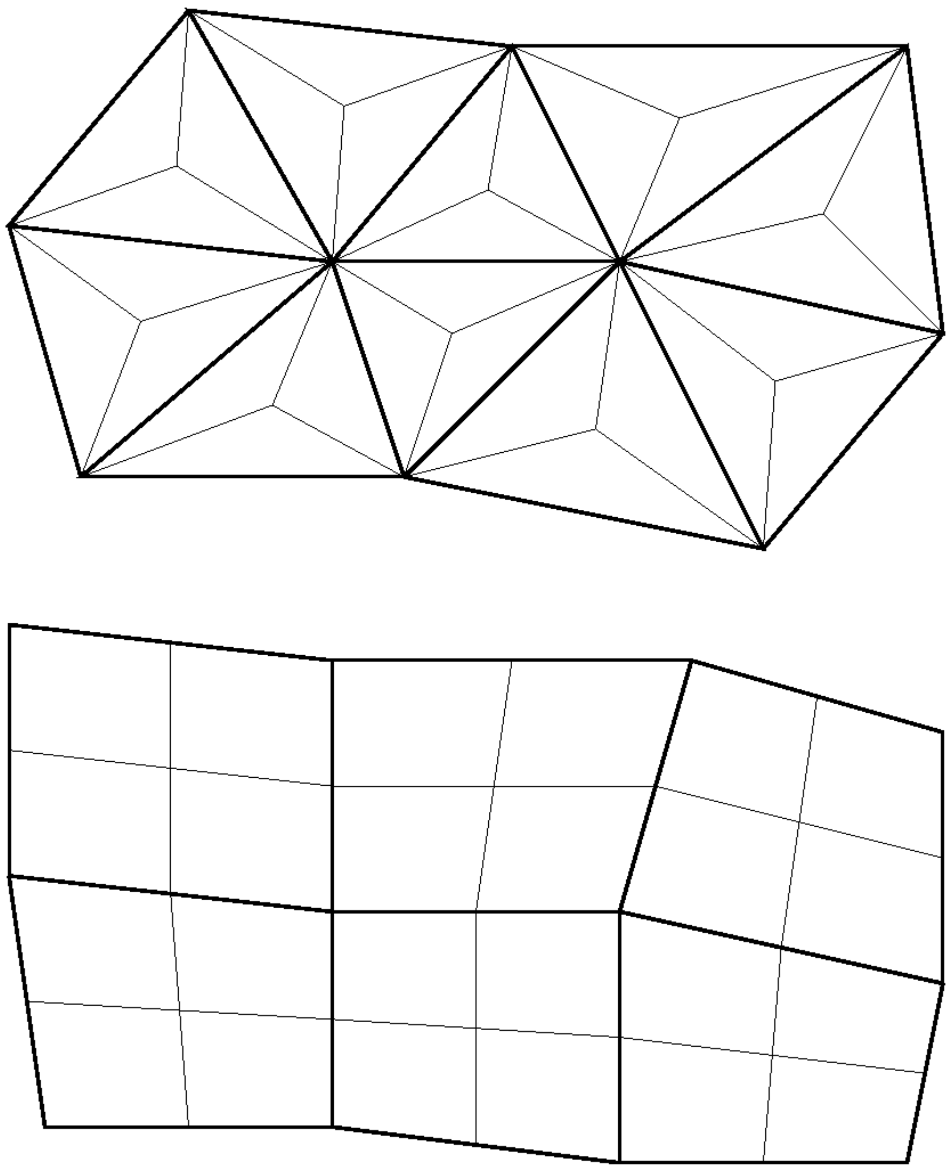
$D_h \subset P_{m,\mathscr{M}_h}$ for some $m \in \mathbb{N}_0$

b_M are generated by a reference bubble function

$\hat{b} \in C(\hat{T}) \cap H_0^1(\hat{T}), \qquad \hat{b} \geq 0, \qquad \hat{b} \neq 0$

simplest choice: $Y_h(M) = b_M \cdot D_h(M)$ (smallest possible dim.)

Meshes for the two–level approach



Residual–based stabilizations (RBS)

The most popular residual–based stabilization is the SUPG method by Brooks, Hughes (1982):

Find $u_h \in W_h$ such that $u_h - \widetilde{u}_{bh} \in V_h$ and

$$a(u_h, v_h) + (R_h(u_h), \delta\, \mathbf{b} \cdot \nabla v_h) = (f, v_h) + \langle g, v_h \rangle_{\Gamma^N} \qquad \forall\, v_h \in V_h\,,$$

where $\quad R_h(u) = -\varepsilon\, \Delta_h u + \mathbf{b} \cdot \nabla u + c\, u - f$

Advantages: robust, easy to implement, accurate away from layers

Drawbacks: non–symmetric, second–order derivatives

LPS: symmetric, operations *discretization* and *optimization* commute

Assumptions and notation

$$W_h \supset P_{k,\mathscr{T}_h} \cap H^1(\Omega) \qquad \text{or} \qquad W_h \supset Q_{k,\mathscr{T}_h} \cap H^1(\Omega)$$

approximation property of κ_h: $\quad \forall\, l \in \{0,\dots,k\}$:

$$\|\kappa_h q\|_{0,M} \le C_\kappa \frac{h_M^l}{k^l} |q|_{l,M} \quad \forall\, q \in L^2(\Omega),\, q|_M \in H^l(M) \;\; \forall\, M \in \mathscr{M}_h$$

inverse inequality: $\quad |v_h|_{1,M} \le \mu_k h_M^{-1} \|v_h\|_{0,M} \quad \forall M \in \mathscr{M}_h,\ \forall\, v_h \in W_h$

$(\mu_k \ge k)$

local projection norm:

$$|||v|||_{LP} = \left(\varepsilon |v|_{1,\Omega}^2 + \|\sigma^{1/2} v\|_{0,\Omega}^2 + \frac{1}{2} \|(\mathbf{b}\cdot\mathbf{n})^{1/2} v\|_{0,\Gamma^N}^2 + s_h(v,v) \right)^{1/2}$$

Then $\quad a_h(v,v) = |||v|||_{LP}^2 \quad \forall\, v \in V \quad \Longrightarrow \quad \exists$ unique u_h

General error estimate

$$|||u-u_h|||_{LP} \lesssim \inf_{w_h \in W_h^b} \left(\|u-w_h\|^2_{0,\Gamma^N} + \sum_{M \in \mathcal{M}_h} C_M \, \|u-w_h\|^2_{1,M,*} \right)^{1/2} + \sup_{v_h \in V_h} \frac{s_h(u,v_h)}{|||v_h|||_{LP}}$$

$$W_h^b = \{w_h \in W_h\, ;\; w_h - \widetilde{u}_{bh} \in V_h\},$$

$$\|w\|_{1,M,*} = |w|_{1,M} + \mu_k \, h_M^{-1} \, \|w\|_{0,M}$$

if s_M are SD–based

$$C_M = (1+\beta^{-1})^2 \, (\lambda_M + h_M^2 \, \mu_k^{-2} + \|\mathbf{b}\|^2_{0,\infty,M} \, h_M^2 \, \mu_k^{-2} \, \lambda_M^{-1}) \,,$$

$$\lambda_M = \max\{\varepsilon, \tau_M \, \|\mathbf{b}\|^2_{0,\infty,M}\} \,.$$

if s_M are gradient–based

$$C_M = (1+\beta^{-1})^2 \, (\lambda_M + h_M^2 + \|\mathbf{b}\|^2_{0,\infty,M} \, h_M^2 \, \mu_k^{-2} \, \lambda_M^{-1}) \,,$$

$$\lambda_M = \max\{\varepsilon, \tau_M\} \,.$$

Main result for the gradient–based LPS

Assume: $u \in H^{l+1}(\Omega)$ for some $l \in \{1, \ldots, k\}$,

$\widetilde{u}_{bh}$ sufficiently accurate,

$$\tau_M \sim \min \left\{ \frac{h_M \, \|\mathbf{b}\|_{0,\infty,M}}{\mu_k}, \frac{h_M^2 \, \|\mathbf{b}\|_{0,\infty,M}^2}{\varepsilon \, \mu_k^2} \right\}$$

Then

$$|||u - u_h|||_{LP} \lesssim \frac{h^{l+1/2}}{k^l} \, \|u\|_{l+1,\Omega} + \frac{h^l}{k^l} \, \frac{\mu_k}{k} \left(1 + \frac{1}{\beta}\right) \left(\varepsilon^{1/2} + h + \frac{h^{1/2}}{\mu_k^{1/2}}\right) \|u\|_{l+1,\Omega}$$

Main result for the SD–based LPS

Assume: $u \in H^{l+1}(\Omega)$ for $l \in \{1,\dots,k\}$, $\widetilde{u}_{bh}$ suff. accurate,

$\mathbf{b}|_M \in W^{l,\infty}(M)^d$ for all $M \in \mathscr{M}_h$,

$$\tau_M \sim \min\left\{ \frac{h_M}{\mu_k \|\mathbf{b}\|_{0,\infty,M}}, \frac{h_M^2}{\varepsilon \mu_k^2} \right\}$$

Then

$$|||u-u_h|||_{LP} \lesssim \frac{h^{l+1/2}}{k^l} \|u\|_{l+1,\Omega}$$
$$+ \frac{h^l}{k^l}\frac{\mu_k}{k}\left(1+\frac{1}{\beta}\right)\left(\varepsilon^{1/2} + \frac{h^{1/2}}{\mu_k^{1/2}}\right)\|u\|_{l+1,\Omega}$$
$$+ \left(\sum_{M\in\mathscr{M}_h} \min\left\{ \frac{h_M \|\mathbf{b}\|^2_{l,\infty,M}}{\mu_k \|\mathbf{b}\|_{0,\infty,M}}, \frac{\|\mathbf{b}\|^2_{l,\infty,M}}{\sigma_0} \right\} \frac{h_M^{2l}}{k^{2l}} \|u\|^2_{l+1,M}\right)^{\frac{1}{2}}$$

Stability in the SUPG norm

K., Tobiska (2008)

$$|||v_h|||_{SUPG} = \left(\varepsilon |v_h|_{1,\Omega}^2 + \sigma_0 \, \|v_h\|_{0,\Omega}^2 + \sum_{M \in \mathscr{M}_h} \delta_M \, \|\mathbf{b} \cdot \nabla v_h\|_{0,M}^2 \right)^{1/2}$$

$$\delta_M \sim \min \left\{ \frac{h_M}{\mu_k \, \|\mathbf{b}\|_{0,\infty,M}}, \frac{h_M^2}{\varepsilon \, \mu_k^2} \right\}$$

Inf–sup condition

We assume that there exists a space $B_h \subset V_h$ such that

$$B_h = \bigoplus_{M \in \mathscr{M}_h} B(M) \qquad \text{with} \qquad B(M) \subset H_0^1(M)\,.$$

Π_M ... orthogonal L^2 projection of $L^2(M)$ onto $B(M)$, $\quad M \in \mathscr{M}_h$

stronger norm

$$|||v||| = \left(|||v|||_G^2 + \sum_{M \in \mathscr{M}_h} \left\{ \delta_M \, \|\Pi_M(\mathbf{b} \cdot \nabla v)\|_{0,M}^2 + \tau_M \, s_M(v,v) \right\} \right)^{1/2}$$

Then

$$\exists\, \beta > 0 : \qquad \sup_{v_h \in V_h} \frac{a_h(u_h, v_h)}{|||v_h|||} \geq \beta\, |||u_h||| \qquad \forall\, u_h \in V_h$$

(β independent of h and ε)

Proof of the inf–sup condition

$$z_h|_M = \delta_M \Pi_M(\mathbf{b}\cdot\nabla u_h) \quad \forall\, M\in\mathcal{M}_h \qquad \Rightarrow \qquad z_h \in B_h \quad \text{and}$$

$$(\mathbf{b}\cdot\nabla u_h, z_h)_M = \delta_M \,\|\Pi_M(\mathbf{b}\cdot\nabla u_h)\|_{0,M}^2 \quad \forall\, M\in\mathcal{M}_h$$

$$\Rightarrow \qquad a_h(u_h,z_h) = \sum_{M\in\mathcal{M}_h} \delta_M \,\|\Pi_M(\mathbf{b}\cdot\nabla u_h)\|_{0,M}^2$$

$$+\,\varepsilon\,(\nabla u_h,\nabla z_h) + (c\,u_h,z_h) + \sum_{M\in\mathcal{M}_h} \tau_M\, s_M(u_h,z_h)$$

$$\geq \frac{1}{2}\sum_{M\in\mathcal{M}_h} \delta_M \,\|\Pi_M(\mathbf{b}\cdot\nabla u_h)\|_{0,M}^2 - \zeta\, a_h(u_h,u_h)$$

$$\Rightarrow \qquad v_h := 2\,z_h + (1+2\,\zeta)\,u_h \quad \text{satisfies} \quad a_h(u_h,v_h) \geq |||u_h|||^2$$

furthermore, $\quad |||z_h||| \leq C\,|||u_h||| \qquad \Rightarrow \qquad |||u_h||| \geq \beta\,|||v_h|||$

$\Rightarrow$ inf–sup condition holds

Relation between the norms $|||\cdot|||$ and $|||\cdot|||_{SUPG}$

Assumptions: $\mathbf{b} \neq \mathbf{0}$ in $\overline{\Omega}$,

all cells of $\mathscr{M}_h$ are affine equivalent to $\widehat{T}$

If s_M are gradient–based and $\mathbf{b}$ is constant or
if s_M are SD–based and $\mathbf{b}$ is piecewise polynomial, then

$$|||v_h||| \geq C\,|||v_h|||_{SUPG} \qquad \forall\, v_h \in W_h\,.$$

Otherwise, there exists $h_0 > 0$ independent of ε such that this inequality holds for $0 < h \leq h_0$. The constant C is positive and independent of h and the data of the problem.

Theorem 1 *There exists* $h_0 > 0$ *independent of* ε *such that for* $0 < h \leq h_0$

$$\tilde{\beta}\, |||u_h|||_{SUPG} \leq \sup_{v_h \in V_h} \frac{a_h(u_h, v_h)}{|||v_h|||_{SUPG}} \qquad \forall\, u_h \in V_h$$

with a positive constant $\tilde{\beta}$ *independent of* h *and* ε*. If* $\mathbf{b}$ *is constant or, in case of SD–based* s_M*, if* $\mathbf{b}$ *is piecewise polynomial, then the inf–sup condition holds for any* h*.*

$\Rightarrow$ The local projection stabilization controls not only the fluctuations but also the streamline derivatives.

$\Rightarrow$ The above convergence results hold also in the SUPG norm.

Relation to residual–based stabilizations

K., Lube (2008)

Assumptions:

– $\operatorname{div}\mathbf{b} = 0$, $c = const.$, $\Gamma^N = \emptyset$, $u_b = 0$

– simplicial triangulations

Then $V_h = \overline{V}_h \oplus B_h$ with $\overline{V}_h := P_{k,\mathscr{M}_h} \cap V$

$$B_h := \bigoplus_{M \in \mathscr{M}_h} B_k(M), \quad B_k(M) \subset H_0^1(M)$$

Gradient–based LPS scheme:

$$a(u_h,v_h)+\sum_{M\in\mathscr{M}_h}\tau_M(\kappa_h\nabla u_h,\kappa_h\nabla v_h)_M=(f,v_h)\qquad\forall\, v_h\in V_h,$$

Note: $\bar{v}_h\in\overline{V}_h \quad\Rightarrow\quad \nabla\bar{v}_h\in[D_h]^d \quad\Rightarrow\quad \kappa_h\nabla\bar{v}_h=\mathbf{0}$

$$u_h=\bar{u}_h+u_h^b,\qquad u_M:=u_h^b|_M\quad\forall\, M\in\mathscr{M}_h$$

$$\underbrace{a_M(u_M,v_M)+\tau_M(\kappa_M\nabla u_M,\kappa_M\nabla v_M)_M}_{(A_M u_M,v_M)_M}=(f-L\bar{u}_h,v_M)_M\;\forall\, v_M\in B_k(M)$$

$$Lu=-\varepsilon\,\Delta u-\mathbf{b}\cdot\nabla u+c\,u$$

$$A_M:B_k(M)\to B_k(M)$$

$$\Rightarrow\quad u_M=A_M^{-1}\rho_M(f-L\bar{u}_h)$$

where ρ_M is the orthogonal L^2 projection from $L^2(M)$ onto $B_k(M)$

Residual–based formulation of LPS

$$a(\overline{u}_h,\overline{v}_h)+\sum_{M\in\mathscr{M}_h}(f-L\overline{u}_h,(A_M^\star)^{-1}\rho_M L^\star\overline{v}_h)_M=(f,\overline{v}_h)\qquad\forall\overline{v}_h\in\overline{V}_h$$

$\sim$ "unusual" GLS method (Franca, Valentin (2000))

Theorem 2 *There exist positive constants* C_1 *and* C_2 *such that, for any* $M\in\mathscr{M}_h$ *and* $g\in B_k(M)$, *we have*

$$\frac{C_1 h_M^2}{\varepsilon+\tau_M+\|\mathbf{b}\|_{[L^\infty(M)]^d}h_M+c\,h_M^2}\le\frac{\|(A_M^\star)^{-1}g\|_{0,M}}{\|g\|_{0,M}}\le\frac{C_2 h_M^2}{\varepsilon+\tau_M+c\,h_M^2}\,.$$

Proof The most difficult part is to show that there exists $\gamma>0$ such that $\quad\|\kappa_M\nabla v\|_{0,M}\ge\gamma\|\nabla v\|_{0,M}\quad\forall v\in B_k(M),M\in\mathscr{M}_h$.

Recovering of the SUPG method for $k = 1$

$\dim B_1(M) = 1 \quad \Rightarrow \quad A_M^\star$ represents a multiplicative factor:

$$(A_M^\star)^{-1} = \frac{\|b_M\|_{0,M}^2}{(\varepsilon + \tau_M)|b_M|_{1,M}^2 + c\,\|b_M\|_{0,M}^2}$$

Define $\quad \mathbf{b}_M = \dfrac{(\mathbf{b}, b_M)_M}{(1, b_M)_M}, \qquad f_M = \dfrac{(f, b_M)_M}{(1, b_M)_M}$

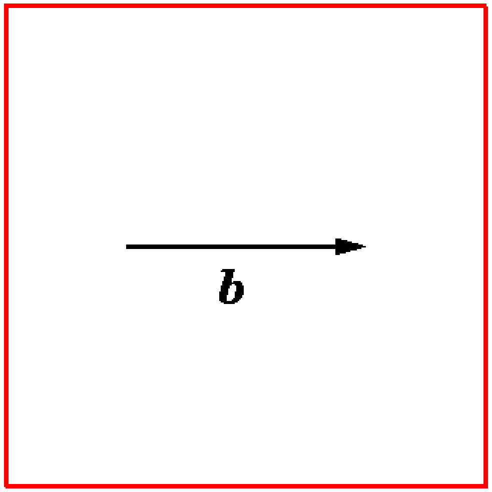
Then

$$\begin{aligned}(f - L\overline{u}_h, (A_M^\star)^{-1}\rho_M L^\star \overline{v}_h)_M \\ = \delta_M(\mathbf{b}_M \cdot \nabla \overline{u}_h + c\,\overline{u}_h - f_M, \mathbf{b}_M \cdot \nabla \overline{v}_h - c\,\overline{v}_h(x_M))_M\end{aligned}$$

with $\quad \delta_M = \dfrac{(1, b_M)_M^2}{|M|\{(\varepsilon + \tau_M)|b_M|_{1,M}^2 + c\,\|b_M\|_{0,M}^2\}}$

Example

$u = 0$



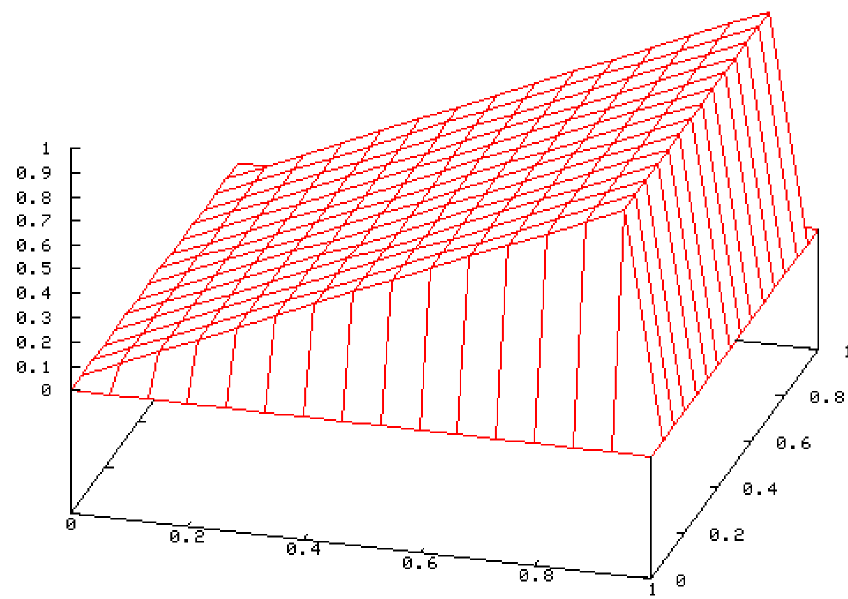
$$\varepsilon = 10^{-8}$$

$$|\mathbf{b}| = 1$$

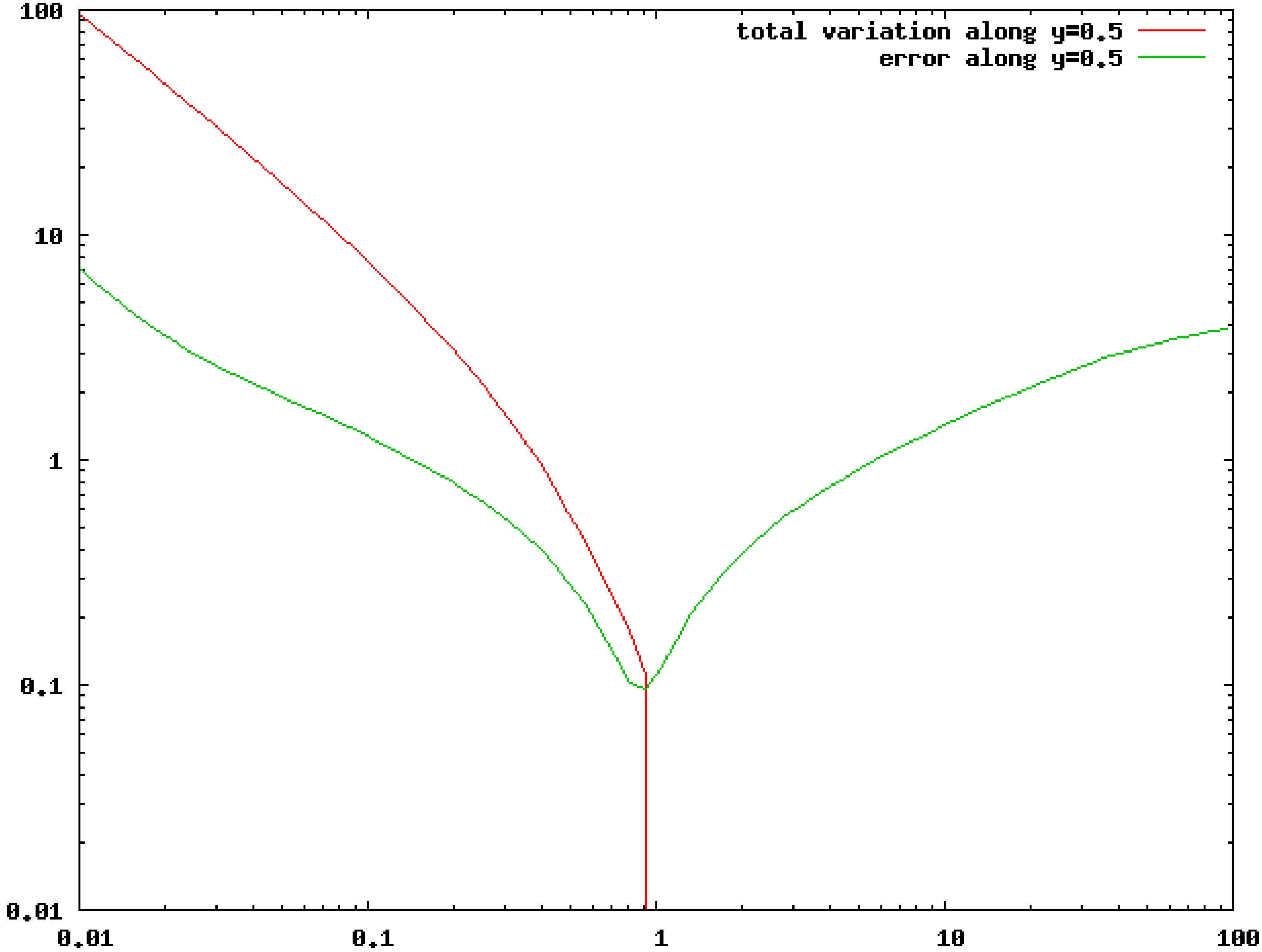
$$f = 1$$

triangulation:

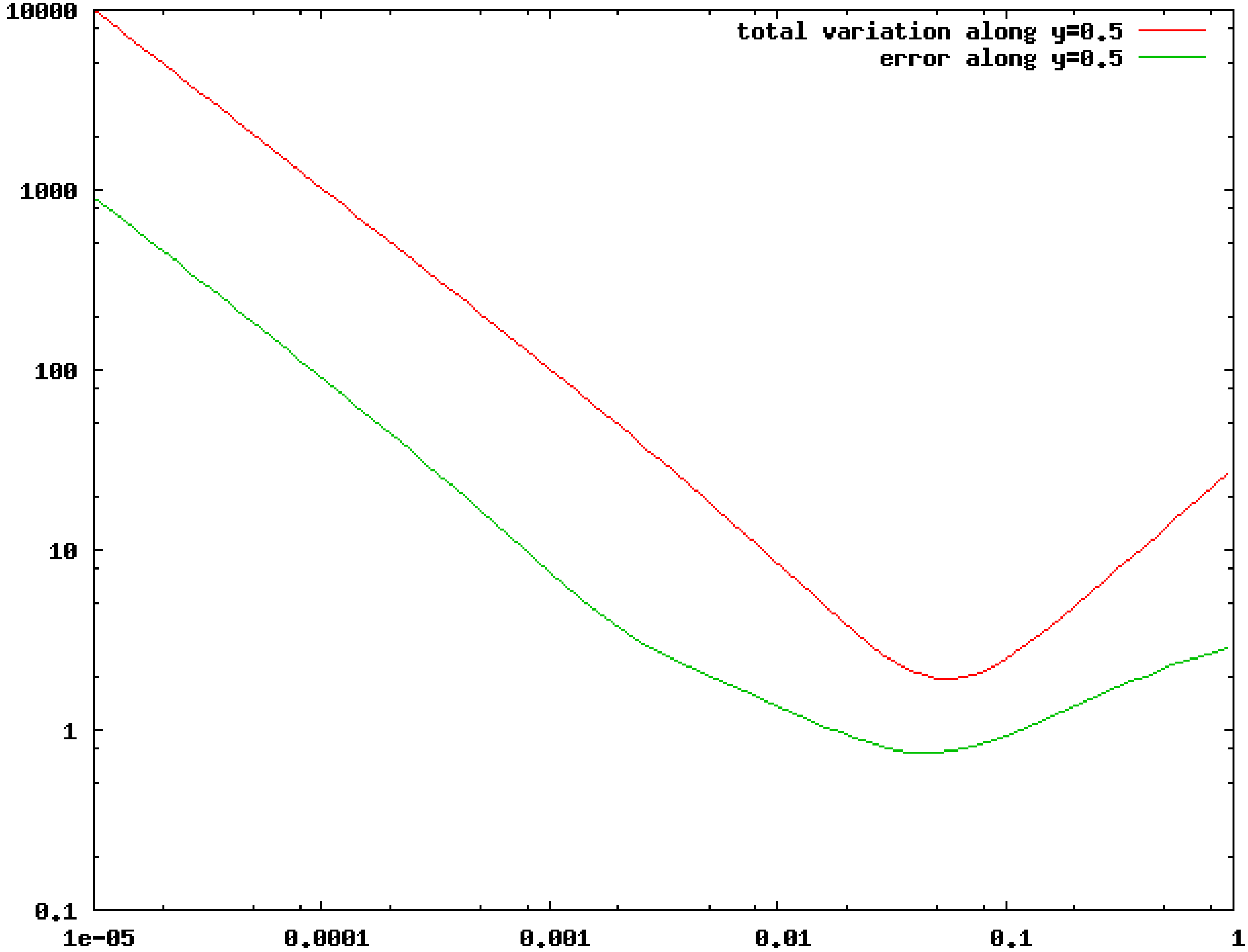
33×33 vertices



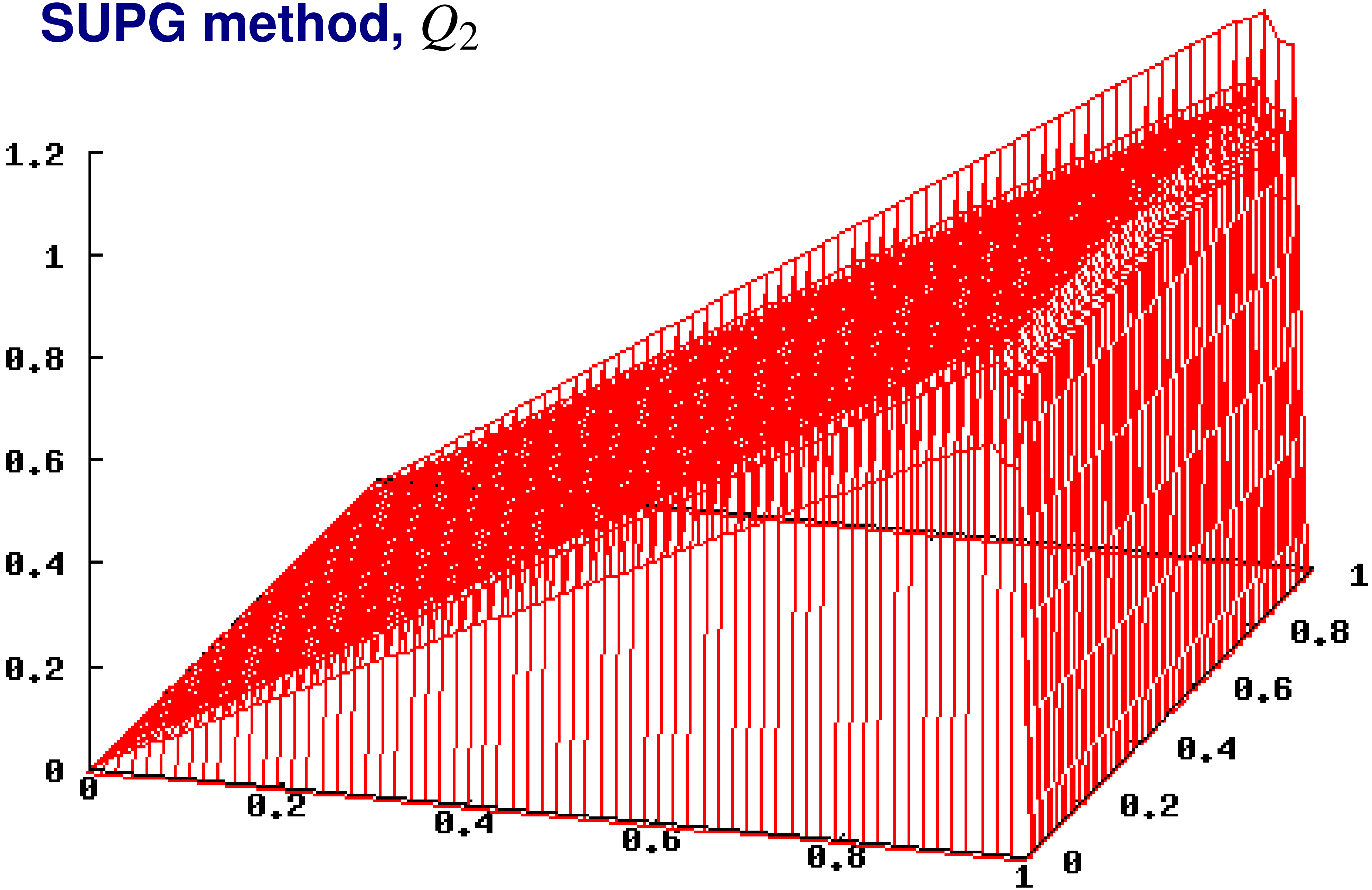
SUPG method, Q_2



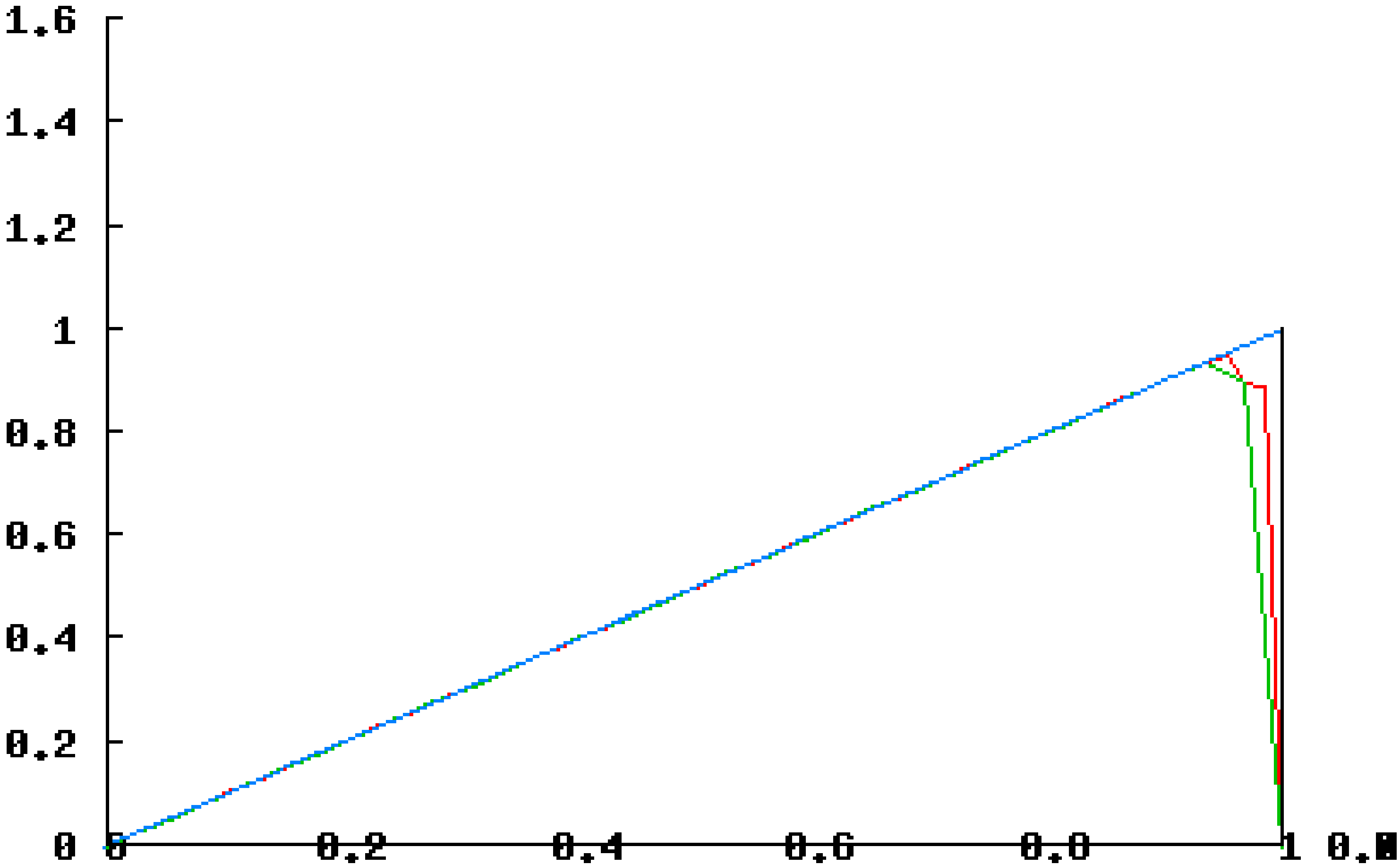
Gradient–based LPS, Q_2^{bub}/P_1^{disc}



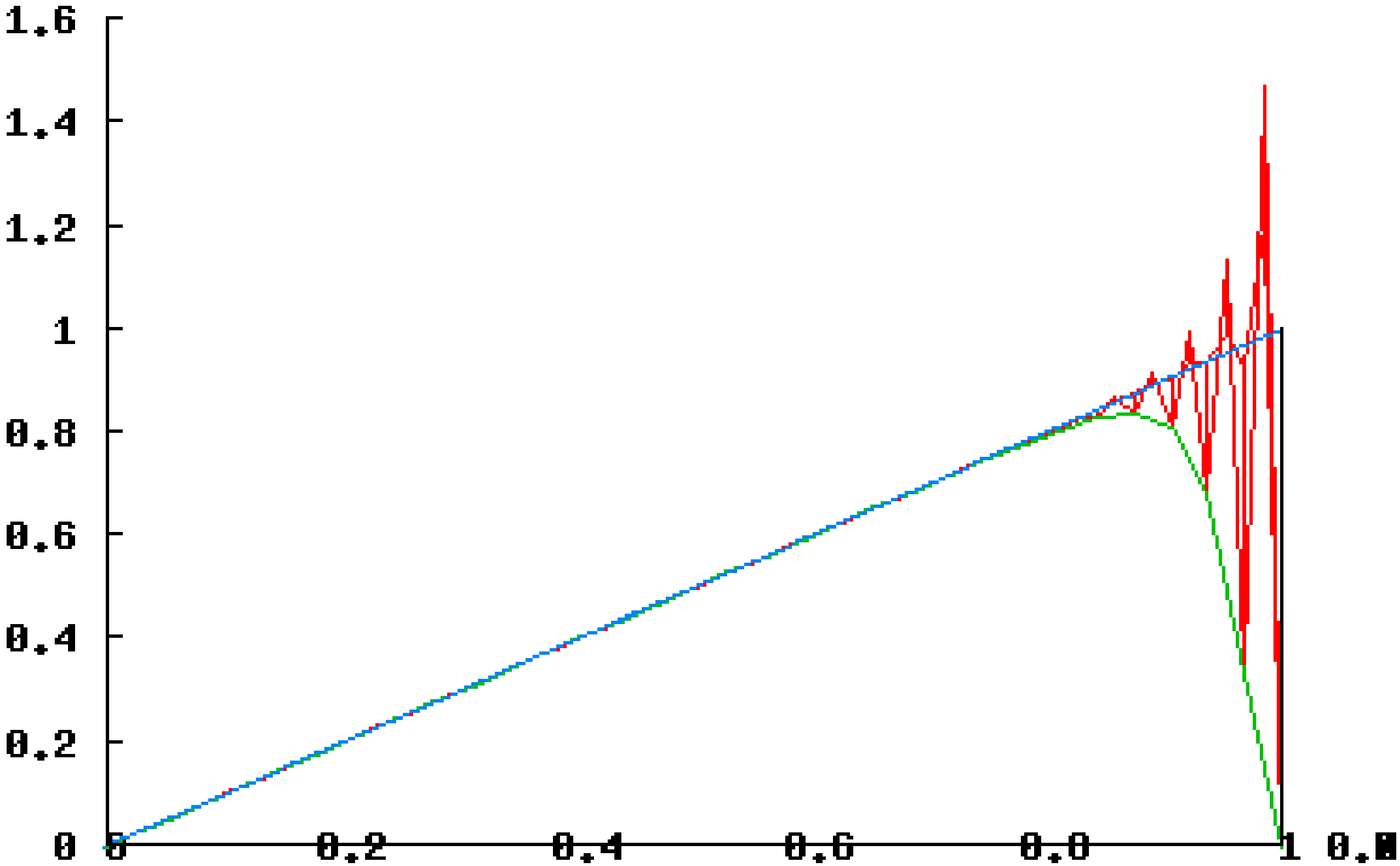
SUPG method, Q_2



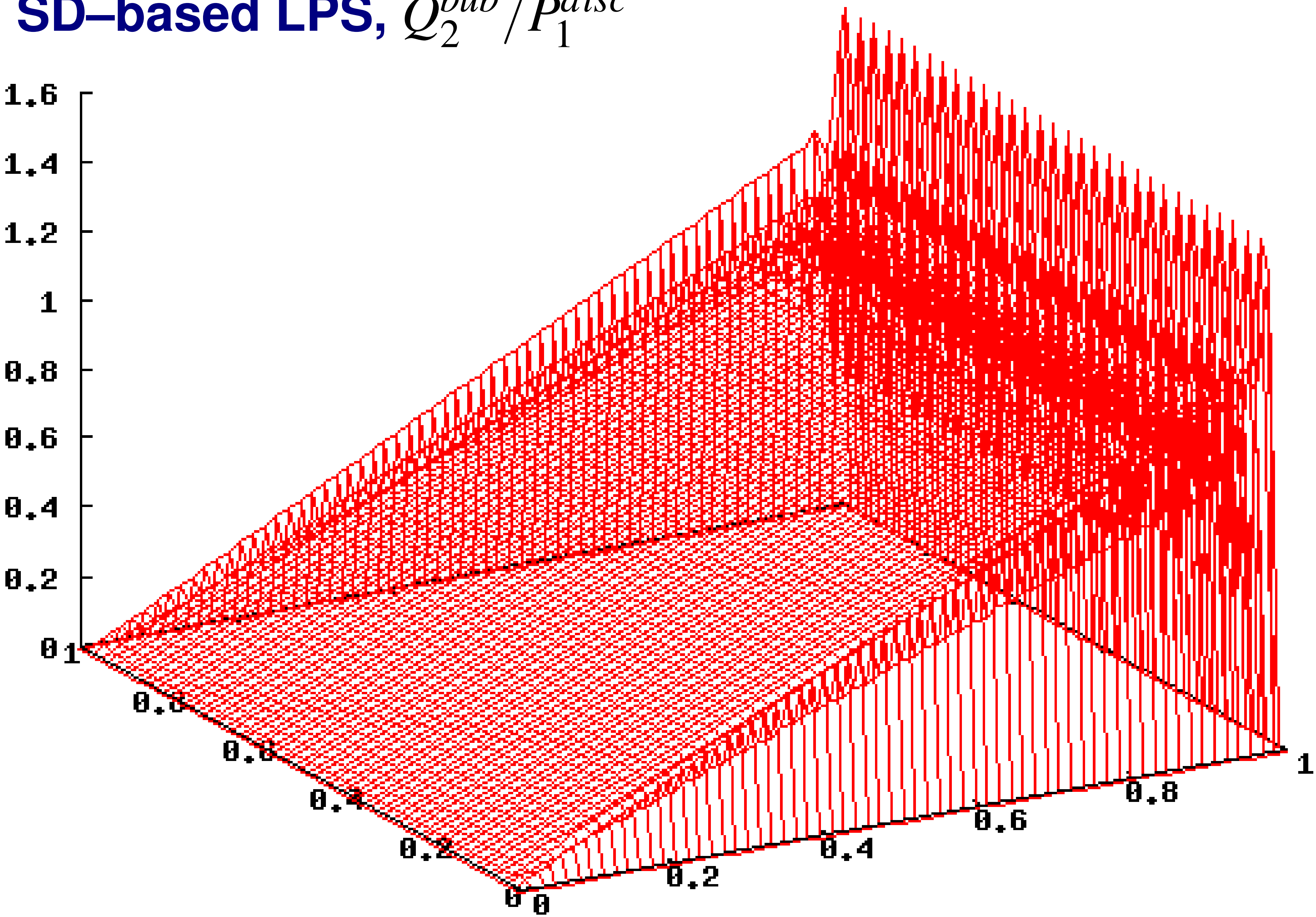
SUPG method, Q_2



SD–based LPS, Q_2^{bub}/P_1^{disc}



SD–based LPS, Q_2^{bub}/P_1^{disc}



Conclusions

– optimal convergence results with respect to h for the LPS applied to convection–diffusion–reaction problems can be obtained for the gradient–based variant and, under additional assumptions, for the SD–based variant

– the LPS methods are more stable than their coercivity suggests

– simplicial LPS methods are closely related to residual–based stabilizations

– LPS methods often do not attain the quality of the SUPG method