The local projection method applied to inf-sup stable discretisations of the Oseen problem

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Contents

Oseen equations

- domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$
- given velocity field b with $\operatorname{div} b = 0$
- Oseen equations with homogeneous Dirichlet b.c.

$$
\begin{aligned}
-\nu \triangle u + (b \cdot \nabla) u + \sigma u + \nabla p &= f && \text{in } \Omega, \\
\operatorname{div} u &= 0 && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Weak formulation of Oseen equations

- spaces: $V := H_0^1(\Omega)^d$, $Q := L_0^2(\Omega)$
- parameters: $\nu > 0$, $\sigma \geq 0$
- $b \in W^{1,\infty}(\Omega)$
- bilinear form

$$A\big((u,p);(v,q)\big) := \nu(\nabla u, \nabla v) + \big((b \cdot \nabla)u, v\big) + \sigma(u,v) - (p, \operatorname{div} v) + (q, \operatorname{div} u)$$

- weak formulation
  Find $(u,p) \in V \times Q$ such that

$$A\big((u,p);(v,q)\big) = (f,v) \qquad \forall (v,q) \in V \times Q$$

- uniquely solvable due to inf-sup condition for (V, Q)

Discrete spaces

- family of shape-regular triangulation $\{\mathcal{T}_h\}$
- discrete spaces
  - velocity $V_h \subset V$: elements of order r
  - pressure $Q_h \subset Q$: elements of order $r-1$
- discrete inf-sup condition for (V_h, Q_h)

$$\exists \beta > 0 \; \forall h \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\operatorname{div} v_h, q_h)}{\|q_h\|_0 \, |v_h|_1} \geq \beta$$

Discrete Problem

- discrete problem without any stabilisation

  Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$A\big((u_h, p_h); (v_h, q_h)\big) = (f, v_h) \qquad \forall (v_h, q_h) \in V_h \times Q_h$$

- generally unstable due to dominating convection

stabilise by local projection methods

Local projection I

- on each $K \in \mathcal{T}_h$:
  - finite dimensional spaces $D_h^1(K)$, $D_h^2(K)$
  - local L^2 projections

$$\pi_K^1 : L^2(K) \to D_h^1(K), \qquad \pi_K^2 : L^2(K) \to D_h^2(K)$$

- projection spaces (discontinuous w.r.t. $\mathcal{T}_h$)

$$D_h^1 = \bigoplus_{K \in \mathcal{T}_h} D_h^1(K), \qquad D_h^2 = \bigoplus_{K \in \mathcal{T}_h} D_h^2(K)$$

- global projections $\pi_h^i : L^2(\Omega) \to D_h^i$, $i = 1, 2$:

$$(\pi_h^i w)|_K := \pi_K^i (w|_K)$$

Local projection II

- fluctuation operators $\kappa_h^i : L^2(\Omega) \to L^2(\Omega)$, $i = 1, 2$:

$$\kappa_h^i = id - \pi_h^i$$

- approximation property of κ_h^i, $i = 1, 2$:

$$\|\kappa_h^i q\|_{0,K} \leq C\, h_K^\ell |q|_{\ell,K} \qquad \forall q \in H^\ell(K)$$

holds for all $K \in \mathcal{T}_h$, $0 \leq \ell \leq r$, provided

$$P_{r-1}(K) \subset D_h(K)$$

- notation: $P_k = \{0\}$ for all $k < 0$

Stabilisation term

- no pressure stabilisation due to inf-sup stable elements

- stabilisation term

$$S_h(u,v) := \sum_{K \in \mathcal{T}_h} \Big(\tau_K \big(\kappa_h^1 (b \cdot \nabla) u, \kappa_h^1 (b \cdot \nabla) v\big)_K + \gamma_K \big(\kappa_h^2 (\operatorname{div} u), \kappa_h^2 (\operatorname{div} v)\big)_K \Big)$$

- user-chosen parameters τ_K, γ_K
- stabilisation of
  - derivative in streamline-direction
  - divergence constraint

Stabilised discrete problem

- bilinear form

$$A_h\big((u,p);(v,q)\big) := \nu(\nabla u, \nabla v) + \big((b\cdot\nabla(u,v)\big) + \sigma(u,v) + S_h(u,v) - (p, \operatorname{div} v) + (q, \operatorname{div} u)$$

- stabilised discrete problem

  Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$A_h\big((u_h,p_h);(v_h,q_h)\big) = (f, v_h) \qquad \forall (v_h,q_h) \in V_h \times Q_h$$

Interpolation error

- norm

$$|||(v,q)||| := \big(\nu|v|_1^2 + \sigma\|v\|_0^2 + (\nu+\sigma)\|q\|_0^2 + S_h(v,v)\big)^{1/2}$$

- remember
  - V_h: elements of order r
  - Q_h: elements of order $r-1$
- interpolation error estimates give

$$|||(u - j_h u, p - i_h p)||| \sim h^r$$

Solvability of stabilised problem

Lemma (Stability) [M., Tobiska 2007]
There exists a positive constant β_2 independent of ν and h such that

$$\inf_{(v_h,q_h)} \sup_{(w_h,r_h)} \frac{A_h\big((v_h,q_h);(w_h,r_h)\big)}{|||(v_h,q_h)|||\ |||(w_h,r_h)|||} \geq \beta_2 .$$

Proof.

- construct for arbitrary pair $(v_h, q_h) \in V_h \times Q_h$ a pair $(w_h, r_h) \in V_h \times Q_h$ such that

$$A_h\big((v_h,q_h);(w_h,r_h)\big) \geq \beta_2 |||(v_h,q_h)|||\ |||(w_h,r_h)|||$$

- use

$$A_h\big((v_h,q_h);(v_h,q_h)\big) = \nu|v_h|_1^2 + \sigma\|v\|_0^2 + S_h\big((v_h,q_h);(v_h,q_h)\big)$$

  and discrete inf-sup condition to control $\|q_h\|_0$

Consistency error

Lemma (Consistency error) [MT07]
Let $s \in [0, r]$, $b|_K \in W^{s,\infty}(K)$, $P_{s-1}(K) \subset D_h^1(K)$

$$\Big| A_h\big((u - u_h, p - p_h); (w_h, r_h)\big) \Big|$$
$$\leq C \left(\sum_{K \in \mathcal{T}_h} \tau_K \, h_K^{2s} \|u\|_{s+1,K}^2 \right)^{1/2} \| (w_h, r_h) \|$$

optimal order $\mathcal{O}(h^r)$ for $\tau_K \lesssim h_K^{2(r-s)}$

Error estimate

Theorem (Error estimate) [MT07]
Let for an s with $0 \leq s \leq r$ the inclusion $P_{s-1}(K) \subset D_h^1(K)$ be fulfilled. Choose $\tau_K \lesssim h_K^{2(r-s)}$ and $\gamma_K \sim 1$. Provided the orthogonality

$$(q - i_h q, \varphi_h) = 0 \qquad \forall \varphi_h \in D_h^2,\ q \in H^2(\Omega) \cap Q$$

is satisfied, the error estimate

$$\left|\!\left|\!\left| (u - u_h, p - p_h) \right|\!\right|\!\right| \leq C_\sigma \left(\sum_{K \in \mathcal{T}_h} h_K^{2r} \left(\|u\|_{r+1,K}^2 + \|p\|_{r,K}^2 \right) \right)^{1/2} \sim h^r$$

holds.

Idea of proof

start with

$$
\begin{aligned}
|||(j_h u - u_h, j_h p - p_h)||| &\le \frac{1}{\beta_2} \sup_{(w_h, r_h)} \frac{A_h\big((j_h u - u_h, j_h p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||} \\
&\le \frac{1}{\beta_2} \sup_{(w_h, r_h)} \frac{A_h\big((u - u_h, p - p_h); (w_h, r_h)\big)}{|||(w_h, r_h)|||} \\
&\quad + \frac{1}{\beta_2} \sup_{(w_h, r_h)} \frac{A_h\big((j_h u - u, j_h p - p); (w_h, r_h)\big)}{|||(w_h, r_h)|||}.
\end{aligned}
$$

- first term: consistency error
- second term: estimate term by term

Estimate of critical terms I

- velocity-pressure coupling I

$$\left|(p - i_h p, \operatorname{div} w_h)\right| = \left|(p - i_h p, \operatorname{div} w_h - \pi_h^2 \operatorname{div} w_h)\right|$$
$$= \left|(p - i_h p, \kappa_h^2 \operatorname{div} w_h)\right|$$
$$\leq C \left(\sum \frac{h_K^{2r}}{\gamma_K} \|p\|_{r,K}^2 \right)^{1/2} |||(w_h, r_h)|||$$

- $(q - i_h q, \varphi_h) = 0$ for all $\varphi_h \in D_h^2$ fulfilled for
  - $D_h^2 = \{0\}$
  - discont. pressure: $D_h^2(K) \subset Q_h|_K + \text{span}(1)$
  - cont. pressure: $D_h^2(K) \subset \left(Q_h|_K + \text{span}(1)\right) \cap H_0^1(K)$
    (bubble part of local pressure space)

Estimate of critical terms II

- convective term

$$\left|\big((b\cdot\nabla)(j_h u - u), w_h\big)\right| \le C\left(\sum h_K^{2r}\|u\|_{r+1,K}^2\right)^{1/2}\|w_h\|_0$$

$$\le C\left(\sum \frac{h_K^{2r}}{\nu+\sigma}\|u\|_{r+1,K}^2\right)^{1/2}|||(w_h, r_h)|||$$

- velocity-pressure coupling II

$$\left|\big(r_h, \operatorname{div}(j_h u - u)\big)\right| \le \|r_h\|_0 \left\|\operatorname{div}(j_h u - u)\right\|_0$$

$$\le C\left(\sum \frac{h_K^{2r}}{\nu+\sigma}\|u\|_{r+1,K}^2\right)^{1/2}|||(w_h, r_h)|||$$

- both estimates are not robust in ν for $\sigma = 0$
- only usual properties of interpolation operator j_h needed

Taylor–Hood family

- simplices $V_h = P_r$, $Q_h = P_{r-1}$, $r \geq 2$

$$D_h^1 = P_{s-1}^{\text{disc}},\ s \leq r, \qquad \tau_K \lesssim h_K^{2(r-s)},$$
$$D_h^2 = P_{t-1}^{\text{disc}},\ t \leq r - d - 1,\ \gamma_K \sim 1$$

- quadrilaterals/hexahedra $V_h = Q_r$, $Q_h = Q_{r-1}$, $r \geq 2$

$$D_h^1 = Q_{s-1}^{\text{disc}},\ s \leq r, \qquad \tau_K \lesssim h_K^{2(r-s)},$$
$$D_h^2 = Q_{t-1}^{\text{disc}},\ t \leq r - 2,\ \gamma_K \sim 1$$

- convergence order $|||\cdot||| = \mathcal{O}(h^r)$

Discontinuous pressure

- simplices $V_h = P_r^+$, $Q_h = P_{r-1}^{\text{disc}}$, $r \geq 2$

$$D_h^1 = P_{s-1}^{\text{disc}},\ s \leq r,\ \tau_K \lesssim h_K^{2(r-s)},$$
$$D_h^2 = P_{t-1}^{\text{disc}},\ t \leq r,\ \gamma_K \sim 1$$

- quadrilaterals/hexahedra $V_h = Q_r$, $Q_h = P_{r-1}^{\text{disc}}$, $r \geq 2$

$$D_h^1 = P_{s-1}^{\text{disc}},\ s \leq r,\ \tau_K \lesssim h_K^{2(r-s)},$$
$$D_h^2 = P_{t-1}^{\text{disc}},\ t \leq r,\ \gamma_K \sim 1$$

- convergence order $|||\cdot||| = \mathcal{O}(h^r)$

Robust estimates

- integration by parts of
  - convective term
  - velocity-pressure term with r_h
- pressure jump terms across edges/faces have to be added for discontinuous pressure spaces Q_h

- observation: above terms give now at least order $\mathcal{O}(h^{r+1/2})$ provided an additional orthogonality holds

- however: convergence order is limited to $\mathcal{O}(h^r)$ by velocity-pressure term with $p - i_h p$ since Q_h consists of element of order $r-1$

- idea: use elements of order r for Q_h

Mini-element family I

- simplicial mesh $\mathcal{T}_h$

- velocity space

$$P_r^{++}(K) := P_r(K) + b \cdot P_{r-1}(K),$$

$$P_r^{++} := \left\{v \in H_0^1(\Omega) \;:\; v|_K \in P_r^{++}(K), \forall K \in \mathcal{T}_h\right\}$$

with lowest order bubble function $b \in P_{d+1}(K)$

- continuous pressure space

$$Q_h := \left\{q \in H^1(\Omega) \;:\; q|_K \in P_r(K), \forall K \in \mathcal{T}_h\right\} \cap L_0^2(\Omega)$$

- equal order approximation but different spaces

Mini-element family II

- discrete inf-sup condition fulfilled
  Fortin operator can be constructed

- $D_h^1 = P_{r-1}^{\text{disc}}$
- $D_h^2 = P_{t-1}^{\text{disc}}$, $t \leq r - d$
- $\gamma_K \sim h_K$, $\tau_K \sim h_K$

- convergence order $\mathcal{O}(h^{r+1/2})$

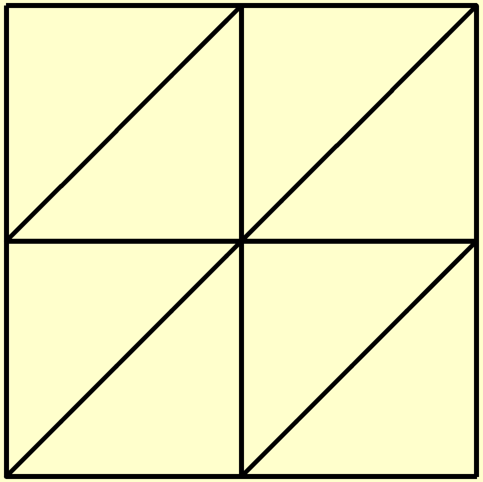
Numerical results

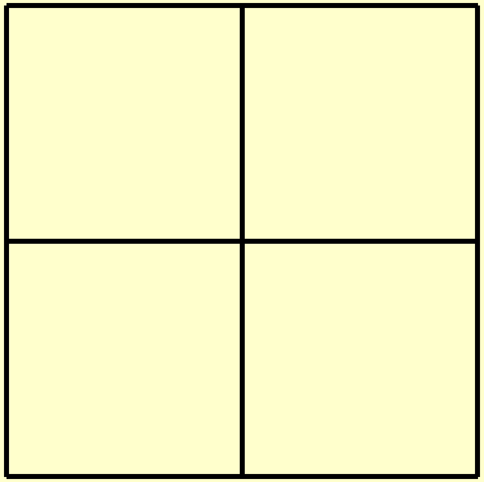
- prescribed solution of problem

$$u(x,y) = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix},$$

$$p(x,y) = 2\cos(x)\sin(y) - p_0 \in L_0^2(\Omega)$$

- $\nu = 10^{-8}$, convection field $b = u$





Mini-Element P_1^{++}/P_1

$$r=1,\, d=2 \quad\Longrightarrow\quad \begin{cases} & D_h^1 = P_0^{\text{disc}},\, \tau_K \sim h_K \\ t \le -1 \quad\Longrightarrow & D_h^2 = \{0\},\quad \gamma_K \sim h_K \end{cases}$$

$$D_h^1 = P_0^{\text{disc}},\, \tau_K = h_K,\, D_h^2 = \{0\},\, \gamma_K = h_K$$

σ	$\lVert u-u_h\rVert_0$	$\lvert u-u_h\rvert_1$	$\lVert p-p_h\rVert_0$	LP-norm	order
10	1.314–6	1.851–3	3.664–6	1.041–4	1.50
1	1.550–6	1.861–3	2.435–6	1.034–4	1.50
0	1.754–6	1.878–3	2.466–6	1.033–4	1.51

Summary

- local projection stabilisation for inf-sup stable pairs
- usual pairs of order r ensure convergence order r
- estimates only robust in ν for $\sigma > 0$
- equal order inf-sup stable approximation
- new pairs robost in ν even for $\sigma = 0$