Taus for systems

R. Codina J. Principe

VMS 2008

Saarbrücken, June 2008

Outline

Outline

Starting point

Consider a BVP problem

$$\mathcal{L}u = f \quad + \text{BC's}$$

with variational form

$$u \in V \quad | \quad B(u,v) = L(v) \quad \forall v \in V$$

The basic idea of the VMS method is to split the unknown u as

$$u = u_h + u', \quad V = V_h \oplus V'$$

where u_h belongs to the finite element space V_h and $u' \in V'$ is the subscale. The way to model it defines the particular VMS approximation.

Problem for the subscales

The subscale u' satisfies

$$B(u_h, v') + B(u', v') = L(v') \quad \forall\, v' \in V'$$

which can be written in abstract form as

$$\langle \mathcal{L}u', v' \rangle = \langle f - \mathcal{L}u_h, v' \rangle \quad + \text{Boundary terms} \quad \forall\, v' \in V'$$

Very often, u' is approximated as

$$u' = \tau P'(f - \mathcal{L}u_h)$$

where P' is a projection onto the space of subscales (bubbles, $V_h^{\perp}$,...)

Application to systems

In the case of systems:

$$u' = \tau r_h, \quad r_h = P'(f - \mathcal{L} u_h)$$
$$u', r_h \in \mathbb{R}^n, \quad \tau \in \mathrm{mat}_{\mathbb{R}}(n, n)$$

The way to obtain τ in this case is completely open. We aim to

- Give a (more or less) systematic way to design τ.
- Consider the possibility of taking τ always a diagonal matrix.
- Apply these concepts to several problems of interest.

Outline

Problem statement

Suppose u' restricted to ∂K, $K \in \mathcal{P}_h$, is known for all K ($u' = 0$ is a possibility). We have to approximate

$$\mathcal{L}u' = r_h \quad \text{in } K + \text{ BC's on } \partial K$$

by

$$u' \approx \tau\, r_h \quad \text{in each } K$$

so that

$$\tau \approx \mathcal{L}^{-1}$$

Approximate/heuristic Fourier analysis

Let us denote the Fourier transform by $\widehat{\ }$. Let k/h be the wave number, with k dimensionless.
Basic heuristic assumption: u' is highly fluctuating, and therefore dominated by high wave numbers k. As a consequence:

- Values of u' on ∂K can be neglected to approximate u' in the interior of K.
- The Fourier transform can be evaluated as for functions vanishing on ∂K (and extended to $\mathbb{R}^d$ by zero).

The Fourier-transformed equation for the subscales will be

$$\hat{\mathcal{L}}(k)\hat{u}'(k) = \hat{r}_h(k)$$

Scaling

Suppose that $\mathcal{L}u = f$ is written in such a way that
$f^{\mathrm{t}}u = \sum_{i=1}^{n} f_i u_i$ is dimensionally well defined.
In general, if $f, g \in \operatorname{range}\mathcal{L}$, and $u, v \in \operatorname{dom}\mathcal{L}$,

$$f^{\mathrm{t}}g = \sum_{i=1}^{n} f_i g_i, \quad u^{\mathrm{t}}v = \sum_{i=1}^{n} u_i v_i$$

may not be dimensionally meaningful.
Let M be a scaling matrix, symmetric, positive-definite and possibly diagonal, that makes the products $f^{\mathrm{t}}Mg$ and $u^{\mathrm{t}}M^{-1}v$ dimensionally consistent. Let also

$$|f|_M^2 = f^{\mathrm{t}}Mf \quad M\text{-norm of } f$$

$$|u|_{M^{-1}}^2 = u^{\mathrm{t}}M^{-1}u \quad M^{-1}\text{-norm of } u$$

$$\|f\|_{L^2_M(K)} = \int_K |f|_M^2$$

Main approximation

We propose to obtain τ by imposing $\|\mathcal{L}\|_{L^2_M(K)} \leq \|\tau^{-1}\|_{L^2_M(K)}$.
We have:

$$\begin{aligned}
\|\mathcal{L}u\|^2_{L^2_M(K)} &= \int_K |\mathcal{L}u|^2_M \mathrm{d}x \\
&\approx \int_{\mathbb{R}^d} |\widehat{\mathcal{L}}(k)\widehat{u}(k)|^2_M \mathrm{d}k \\
&\leq \int_{\mathbb{R}^d} |\widehat{\mathcal{L}}(k)|^2_M |\widehat{u}(k)|^2_M \mathrm{d}k \\
&= |\widehat{\mathcal{L}}(k^0)|^2_M \int_{\mathbb{R}^d} |\widehat{u}(k)|^2_M \mathrm{d}k \\
&\approx |\widehat{\mathcal{L}}(k^0)|^2_M \|u\|^2_{L^2_M(K)}
\end{aligned}$$

Our proposal

From the previous approximation, $\|\mathcal{L}\|_{L^2_M(K)} \leq |\widehat{\mathcal{L}}(k^0)|_M$.

Our proposal is to choose τ such that $|\widehat{\mathcal{L}}(k^0)|_M = |\tau^{-1}|_M$. In particular, if

$$\lambda_{\max}(k^0) = \max \operatorname{spec}_{M^{-1}}(\widehat{\mathcal{L}}(k^0)^* M \widehat{\mathcal{L}}(k^0))$$

with $\lambda \in \operatorname{spec}_{M^{-1}} A \iff \exists x \mid Ax = \lambda M^{-1} x$, we will require that $\tau^{-1} M \tau^{-1} = \lambda_{\max} M^{-1}$, that is to say

Design condition

$$M\tau^{-1} = \lambda_{\max}^{1/2}(k^0) I \iff \tau = \lambda_{\max}^{-1/2}(k^0) M$$

The components of k^0 have to be understood as algorithmic constants.

CDR systems

Suppose that

$$\mathcal{L}u = -\partial_p K_{pq} \partial_q u + A_p \partial_p u + Su$$
$$K_{pq}, A_p, S \in \mathrm{mat}_{\mathbb{R}}(n, n)$$

Then

$$\widehat{\mathcal{L}}(k) = k_p k_q K_{pq} + \mathrm{i} k_p A_p + S$$
$$\widehat{\mathcal{L}}(k)^* = k_p k_q K_{pq}^{\mathrm{t}} - \mathrm{i} k_p A_p^{\mathrm{t}} + S^{\mathrm{t}}$$

In any case

$$\mathrm{spec}_{M^{-1}}(\widehat{\mathcal{L}}(k^0)^* M \widehat{\mathcal{L}}(k^0)) \subset \mathbb{R}^+$$

Outline

Problem statement

Differential form:

$$-\nabla \cdot \boldsymbol{\sigma} + \nabla p = \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{1}{2\mu}\boldsymbol{\sigma} - \nabla^S \boldsymbol{u} = \boldsymbol{0}$$

Variational form:

Find $u = (\boldsymbol{u}, p, \boldsymbol{\sigma}) \in V = (H_0^1(\Omega))^d \times L^2(\Omega)/\mathbb{R} \times (L^2(\Omega))^{d\times d}_{\mathrm{sym}}$ such that

$$B(u, v) = L(v) \quad \forall v \in V$$

$$B(u, v) := (\nabla^S \boldsymbol{v}, \boldsymbol{\sigma}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}) + \frac{1}{2\mu}(\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\nabla^S \boldsymbol{u}, \boldsymbol{\tau})$$

$$L(v) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle$$

Stabilized finite element method

Neglecting interelement boundary terms, the stabilized finite element problem is

$$B(u_h, v_h) + (\nabla^S \boldsymbol{v}_h, \boldsymbol{\sigma}') - (p', \nabla \cdot \boldsymbol{v}_h) + \frac{1}{2\mu}(\boldsymbol{\sigma}', \boldsymbol{\tau}_h) = L(v_h)$$

where the subscales are solution of

$$-\nabla \cdot \boldsymbol{\sigma}' + \nabla p' = \boldsymbol{r}_u := P'(\boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma}_h - \nabla p_h)$$
$$\nabla \cdot \boldsymbol{u}' = r_p := P'(-\nabla \cdot \boldsymbol{u}_h)$$
$$\frac{1}{2\mu}\boldsymbol{\sigma}' - \nabla^S \boldsymbol{u}' = \boldsymbol{r}_\sigma := P'(-\frac{1}{2\mu}\boldsymbol{\sigma}_h + \nabla^S \boldsymbol{u}_h)$$

Approximation to the subscales I

Let us consider $u = (u_1, u_2, p, \sigma_{11}, \sigma_{12}, \sigma_{22})$ $(d = 2)$. The first point is to choose matrix M. If $[\cdot]$ denotes a dimensional group:

$$[\boldsymbol{r}_u]^2 \left[\frac{h^2}{\mu^2}\right] = [r_p]^2 = [\boldsymbol{r}_\sigma]^2, \quad [\boldsymbol{u}']^2 \left[\frac{\mu^2}{h^2}\right] = [p']^2 = [\boldsymbol{\sigma}']^2$$

We may take

$$M = \operatorname{diag}(m, m, 1, 1, 1, 1), \quad m := \frac{h^2}{\mu^2}$$

Approximation to the subscales II

Let us consider matrix τ of the form

$$\tau = \text{diag}(\tau_u, \tau_u, \tau_p, \tau_\sigma, \tau_\sigma, \tau_\sigma)$$

We will show that τ_u, τ_p and τ_σ are uniquely determined by dimensionality.

It can be checked that the eigenvalue of the problem

$$M\widehat{\mathcal{L}}(k^0)^{\mathrm{t}} M\widehat{\mathcal{L}}(k^0)x = \lambda x,$$

has dimensions $[\lambda] = [\mu]^{-2}$, and therefore

$$M\tau^{-1}M\tau^{-1} = \text{diag}\left(\tau_u^{-2}m^2, \tau_u^{-2}m^2, \tau_p^{-2}, \tau_\sigma^{-2}, \tau_\sigma^{-2}, \tau_\sigma^{-2}\right)$$

has to have all the diagonal entries of dimension $[\mu]^{-2}$.

Approximation to the subscales III

Being μ the only parameter of the equation, this immediately implies that

Taus for the three field Stokes problem

$$\tau_u = \alpha_u \frac{h^2}{\mu}, \quad \tau_p = \alpha_p 2\mu, \quad \tau_\sigma = \alpha_\sigma 2\mu$$

where α_u, α_p and α_σ are dimensionless constants that play the role of the algorithmic parameters of the formulation.
The subscales are given by

$$\begin{aligned} \boldsymbol{u}' &= \alpha_u \frac{h^2}{\mu} \boldsymbol{r}_u \\ p' &= \alpha_p 2\mu r_p \\ \boldsymbol{\sigma}' &= \alpha_\sigma 2\mu \boldsymbol{r}_\sigma \end{aligned}$$

Outline

Problem statement

Differential equations:

$$\begin{aligned} \partial_t \eta + H\nabla \cdot \boldsymbol{u} + \varepsilon \boldsymbol{u}_0 \cdot \nabla \eta &= f_\eta \\ \partial_t \boldsymbol{u} + g\nabla \eta + \varepsilon \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u} &= \boldsymbol{f}_u \end{aligned}$$

Convection matrices:

$$A_1 = \begin{bmatrix} \varepsilon u_{0,1} & H & 0 \\ g & \varepsilon u_{0,1} & 0 \\ 0 & 0 & \varepsilon u_{0,1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \varepsilon u_{0,2} & 0 & H \\ 0 & \varepsilon u_{0,2} & 0 \\ g & 0 & \varepsilon u_{0,2} \end{bmatrix}$$

Scaling

The differential equations need to be scaled prior to writing the variational from of the problem. The scaling matrix may be taken as

$$S = \begin{bmatrix} \frac{g}{H} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case:

$$v^{t}Sf = \frac{g}{H}\xi f_{\eta} + vf_{u}, \quad [\frac{g}{H}\xi f_{\eta}] = [vf_{u}] = L^{2}T^{-3}$$

$$f^{t}Sf = \frac{g}{H}f_{\eta}^{2} + f_{u}^{2}, \quad [\frac{g}{H}f_{\eta}^{2}] = [f_{u}^{2}] = L^{2}T^{-4}$$

Therefore $M = I$ once the equations have been scaled.

Stabilization parameters

The spectrum of the scaled differential operator is

$$\mathrm{spec}_S\left(\widehat{\mathcal{L}}(k^0)^* S \widehat{\mathcal{L}}(k^0)\right) =$$

$$\left\{\left(\varepsilon(k^0 \cdot u_0) + \sqrt{gH}|k^0|\right)^2, \varepsilon^2(k^0 \cdot u_0)^2, \left(\varepsilon(k^0 \cdot u_0) - \sqrt{gH}|k^0|\right)^2\right\}$$

If we take $\tau = \mathrm{diag}(\tau_\eta, \tau_u, \tau_u)$ then

$$\mathrm{spec}_S(\tau^{-1} S \tau^{-1}) = \{\tau_\eta^{-2}, \tau_u^{-2}, \tau_u^{-2}\}$$

from where

Taus for the shallow water waves

$$\tau_\eta = \tau_u = \frac{h}{C_1 \varepsilon |\boldsymbol{u}_0| + C_2\sqrt{gH}}$$

Stabilized formulation

The final formulation is

$$
\begin{aligned}
0 = {} & \frac{g}{H}(\partial_t \eta_h, \xi_h) - g(\boldsymbol{u}_h, \nabla \xi_h) - \frac{g}{H}(\varepsilon \boldsymbol{u}_0 \eta_h, \nabla \xi_h) - \frac{g}{H}(f_\eta, \xi_h) \\
& + (\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + g(\nabla \eta_h, \boldsymbol{v}_h) + (\varepsilon \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}_h) - (\boldsymbol{f}_u, \boldsymbol{v}_h) \\
& + \tau \frac{g}{H}(P'(\partial_t \eta_h + H \nabla \cdot \boldsymbol{u}_h + \varepsilon \boldsymbol{u}_0 \cdot \nabla \eta_h - f_\eta), H \nabla \cdot \boldsymbol{v}_h + \varepsilon \boldsymbol{u}_0 \cdot \nabla \xi_h) \\
& + \tau (P'(\partial_t \boldsymbol{u}_h + g \nabla \eta_h + \varepsilon \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_h - \boldsymbol{f}_u), g \nabla \xi_h + \varepsilon \boldsymbol{u}_0 \cdot \nabla \boldsymbol{v}_h)
\end{aligned}
$$

In blue: Galerkin terms
In red: stabilization terms
In green: scaling coefficients

Outline

Problem statement

Differential equations:

$$
\begin{aligned}
-\nu \Delta \boldsymbol{u} + \sigma \boldsymbol{u} + \nabla p &= \boldsymbol{f} \\
\nabla \cdot \boldsymbol{u} &= g
\end{aligned}
$$

Variational form:

$$
B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = L([\boldsymbol{v}, q]) \qquad \forall [\boldsymbol{v}, q]
$$

where

$$
\begin{aligned}
B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) &= \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \sigma(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}) \\
L([\boldsymbol{v}, q]) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \langle g, q \rangle
\end{aligned}
$$

Stabilized finite element problem

The final discrete stabilized problem is:

$$B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_s([\boldsymbol{v}_h, q_h])$$

where:

$$\begin{aligned} B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &+ \tau_p \sum_K \langle \nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_h\rangle_K \\ &+ \tau_u \sum_K \langle -\nu\Delta\boldsymbol{u}_h + \sigma\boldsymbol{u}_h + \nabla p_h, \nu\Delta\boldsymbol{v}_h - \sigma\boldsymbol{v}_h + \nabla q_h\rangle_K \\ &+ \tau_f \sum_E \langle [\![\boldsymbol{n}p_h - \nu\partial_n\boldsymbol{u}_h]\!], [\![\boldsymbol{n}q_h + \nu\partial_n\boldsymbol{v}_h]\!]\rangle_E \\ L_s([\boldsymbol{v}_h, q_h]) &= L([\boldsymbol{v}_h, q_h]) \\ &+ \tau_p \sum_K \langle g, \nabla\cdot\boldsymbol{v}_h\rangle_K + \tau_u \sum_K \langle \boldsymbol{f}, \nu\Delta\boldsymbol{v}_h - \sigma\boldsymbol{v}_h + \nabla q_h\rangle_K \end{aligned}$$

Scaling

Noting the dimensional relationships:

$$[\boldsymbol{f}] = [\frac{\nu}{\ell^2} + \sigma][\boldsymbol{u}] + \frac{1}{[\ell]}[p]$$

$$[g] = \frac{1}{[\ell]}[u]$$

we may take as scaling matrix

$$M = \text{diag}(m_u \boldsymbol{I}_3, m_p)$$

$$m_u = \left(\frac{\nu}{\ell^2} + \sigma\right)^{-1}, \quad m_p = \left(\frac{\nu}{\ell^2} + \sigma\right)\ell^2$$

which satisfies

$$[\boldsymbol{f}]^2 m_u = [g]^2 m_p = [\boldsymbol{u}]^2 m_u^{-1} = [p]^2 m_p^{-1}$$

Stabilization parameters

Let $\tau = \text{diag}(\tau_u, \tau_u, \tau_p)$ (in 2D). Imposing the design condition

$$\tau = \lambda_{\max}^{-1/2}(k^0)M$$

it is found that

Taus for the Stokes-Darcy problem

$$\tau_p = c_1 \nu \frac{h^2}{\ell^2} + c_2 \sigma \ell h$$

$$\tau_u = (c_1 \nu + c_2 \sigma \ell h)^{-1} h^2$$

It can be argued that $\tau_f = \tau_u / h$, that is to say:

$$\tau_f = (c_1 \nu + c_2^u \sigma \ell h)^{-1} h$$

Convergence in the Darcy limit (Badia and Codina)

Method $\ell =$	A h	B L_0	C L_0^2/h
$\|e_u\|$ Original	$h^{k+1}+h^l$ Suboptimal	$h^{k+1/2}+h^{l+1/2}$ Quasi-optimal	h^k+h^{l+1} Suboptimal
$\|e_u\|$ Via duality	$h^{k+1}+h^l$ Suboptimal	$h^{k+1}+h^{l+1}$ Optimal	h^k+h^{l+1} Suboptimal
$\|e_p\|$ Original	$h^{k+1}+h^l$ Suboptimal	$h^{k+1/2}+h^{l+1/2}$ Quasi-optimal	h^k+h^{l+1} Suboptimal
$\|e_p\|$ Via duality	$h^{k+2}+h^{l+1}$ Optimal	$h^{k+1}+h^{l+1}$ Optimal	h^k+h^{l+1} Suboptimal
$\|\nabla\cdot e_u\|$	h^k+h^{l-1} Suboptimal	h^k+h^l Optimal	h^k+h^{l+1} Optimal
$\|\nabla e_p\|$	$h^{k+1}+h^l$ Optimal	h^k+h^l Optimal	$h^{k-1}+h^l$ Suboptimal
$k,\ l$ Optimal	$k+1=l$	$k=l$	$k=l+1$

Outline

Problem statement

Differential equations:

$$\boldsymbol{a}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p + \frac{1}{\mu_m\rho}\boldsymbol{b}\times(\nabla\times\boldsymbol{B}) = \boldsymbol{f}_u$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$-\nabla\times(\boldsymbol{u}\times\boldsymbol{b}) + \frac{1}{\mu_m\sigma}\nabla\times\nabla\times\boldsymbol{B} + \nabla r = \boldsymbol{f}_B$$

$$\nabla\cdot\boldsymbol{B} = 0$$

Variational form:

$$\begin{aligned} B(u,v) = {} & (\boldsymbol{v},\boldsymbol{a}\cdot\nabla\boldsymbol{u}) + \nu(\nabla\boldsymbol{v},\nabla\boldsymbol{u}) - (p,\nabla\cdot\boldsymbol{v}) + (q,\nabla\cdot\boldsymbol{u}) \\ & + \frac{1}{\mu_m\rho}(\boldsymbol{B},\nabla\times(\boldsymbol{v}\times\boldsymbol{b})) - \frac{1}{\mu_m\rho}(\boldsymbol{C},\nabla\times(\boldsymbol{u}\times\boldsymbol{b})) \\ & + \frac{1}{\mu_m\rho}\frac{1}{\mu_m\sigma}(\nabla\times\boldsymbol{C},\nabla\times\boldsymbol{B}) + \frac{1}{\mu_m\rho}(\nabla r,\boldsymbol{C}) - \frac{1}{\mu_m\rho}(\nabla s,\boldsymbol{B}) \end{aligned}$$

Stabilization terms

The following terms have to be added to the Galerkin ones (ASGS formulation):

$$\tau_1(\boldsymbol{a}\cdot\nabla\boldsymbol{v}+\nu\Delta\boldsymbol{v}+\nabla q+\frac{1}{\mu_m\rho}\boldsymbol{b}\times(\nabla\times\boldsymbol{C}),$$

$$\boldsymbol{a}\cdot\nabla\boldsymbol{u}-\nu\Delta\boldsymbol{v}+\nabla p+\frac{1}{\mu_m\rho}\boldsymbol{b}\times(\nabla\times\boldsymbol{B}))$$

$$+\tau_2(\nabla\cdot\boldsymbol{v},\nabla\cdot\boldsymbol{u})$$

$$+\tau_3(-\frac{1}{\mu_m\rho}\nabla\times(\boldsymbol{v}\times\boldsymbol{b})-\frac{1}{\mu_m\rho}\frac{1}{\mu_m\sigma}\nabla\times\nabla\times\boldsymbol{C}+\frac{1}{\mu_m\rho}\nabla s,$$

$$-\frac{1}{\mu_m\rho}\nabla\times(\boldsymbol{u}\times\boldsymbol{b})+\frac{1}{\mu_m\rho}\frac{1}{\mu_m\sigma}\nabla\times\nabla\times\boldsymbol{B}+\frac{1}{\mu_m\rho}\nabla r)$$

$$+\tau_4(\frac{1}{\mu_m\rho}\nabla\cdot\boldsymbol{C},\frac{1}{\mu_m\rho}\nabla\cdot\boldsymbol{B})$$

Stabilization parameters

Multiplying the equations for $\boldsymbol{B}$ by $\frac{1}{\mu_m \rho}$, introducing a scaling matrix M and applying the design condition $\tau = \lambda_{\max}^{-1/2}(k^0)M$ it is found that

Taus for the MHD problem

$$\tau_1 = \left(\alpha + \sqrt{\frac{\alpha}{\gamma}}\beta\right)^{-1}, \quad \tau_2 = h^2 \tau_1^{-1}$$

$$\tau_3 = \left(\gamma + \sqrt{\frac{\gamma}{\alpha}}\beta\right)^{-1} (\mu_m \rho)^2, \quad \tau_4 = h^2 \tau_3^{-1}$$

where

$$\alpha := \frac{a}{h} + \frac{\nu}{h^2}, \quad \beta := \frac{1}{\mu_m \rho} \frac{b}{h}, \quad \gamma := \frac{1}{\mu_m \rho} \frac{1}{\mu_m \sigma} \frac{1}{h^2}$$

Outline

Summary and conclusions

The process we have proposed consists of:

- Scaling the differential equations with a (diagonal) matrix M so that $f^{t}Mg$ and $u^{t}M^{-1}v$ are dimensionally consistent.
- Fourier transforming the differential operator to obtain $\widehat{\mathcal{L}}$.
- Choosing τ diagonal.
- Applying the design condition:

$$\tau = \lambda_{\max}^{-1/2}(k^{0})M$$

This process has been applied to several problems of interest (three field formulation of the Stokes problem, waves in shallow waters, Stokes-Darcy problem, MHD problem). In all cases, the resulting finite element formulation turns out to be stable and optimally convergent in appropriate norms.

THANK YOU!