Suitability/Entropy/LES

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Outline

1. BASIC FACTS ABOUT THE NSE

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2. GALERKIN APPROX IN TORUS
3. GALERKIN APPROX + DIRICHLET
4. ARE SUITABLE SOLUTIONS USEFUL?

Outline

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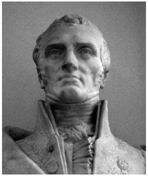
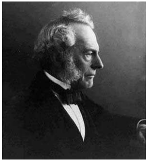


Claude Louis Marie Henri Navier



George Gabriel Stokes

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- u_0 is the initial data.
- f a source term.
- ρ is chosen equal to unity.
- ν is viscosity (inverse of Reynolds number).

EXISTENCE

- J. Leray (1934): introduces the notion of turbulent solution.
  A turbulent solution is a weak solution in
  $u \in L^2(0, T; \mathbf{H}^1(\Omega)) \cap L^\infty(0, T; \mathbf{L}^2(\Omega))$
  $+$ global energy inequality.

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- J. Leray uses mollification to prove existence: $\psi \in D(\mathbb{R}^3)$, $\psi \geq 0$, $\int_{\mathbb{R}^3} \psi = 1$, $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon}\right)$.

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- E. Hopf (1951) *et al.* uses the Galerkin technique to prove existence.

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$\Rightarrow$ Clay Institute 1M$ prize.

SUITABLE WEAK SOLUTION

Definition (V. Scheffer (1976))

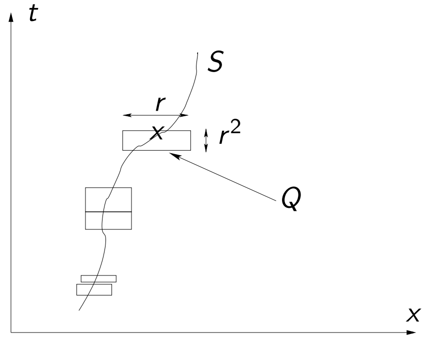
A NS weak solution is said to be suitable weak solutions iff (u, p) is a weak solution and

$$\partial_t(\tfrac{1}{2}u^2) + \nabla\cdot(u(\tfrac{1}{2}u^2 + p)) - \nu\nabla^2(\tfrac{1}{2}u^2) + \nu(\nabla u)^2 - f\cdot u \leq 0.$$

in $\mathcal{D}'((0,T)\times\Omega)$

SUITABLE WEAK SOLUTION

- Singular set: S
  $S = \{(x,t) \in \Omega \times]0,T[,\ \ u \notin L^\infty(V), \forall V \text{ s.t. } (x,t) \in V\}$



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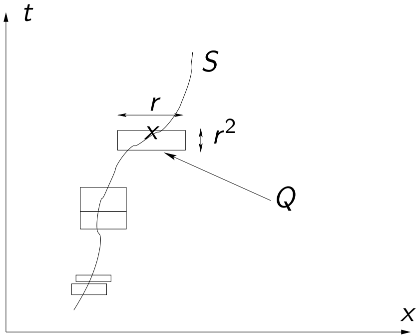

- $\mathcal{P}^1(S) = \lim_{\delta\to 0^+} \inf\{\sum r_i^1,\ S \subset \cup Q(M_i,r_i),\ r_i < \delta\}$

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Best partial regularity theorem to date.

EX 1: CONSTRUCTION BY MOLLIFICATION

- Leray's mollification

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Theorem (Leray (1934), Duchon-Robert (2000))

Unique weak solution for all $t > 0$ if $\alpha > \frac{d+2}{4}$, and $u_\epsilon \underset{\epsilon\to 0}{\longrightarrow} u$ (up to subsequences) and u is suitable.

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- Set $N_\varepsilon = \frac{1}{\varepsilon}$
- Set $\mathbf{X}_\varepsilon = \dot{\mathbb{P}}_{N_\varepsilon}$ (velocity space).
- Let $P_\varepsilon : \dot{\mathbf{L}}^2(\Omega) \longrightarrow \mathbf{X}_\varepsilon$ be L^2-projection.

$$\dot{\mathbf{L}}^2(\Omega) = \mathbf{X}_\varepsilon \oplus (\mathbf{X}_\varepsilon)^\perp$$

EX 2: CONSTRUCTION BY MOLLIFICATION/NLGM

- Solve for u_ϵ, p_ϵ s.t.

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Theorem

For all $\epsilon > 0$, problem is well-posed (existence + uniqueness). $u_\epsilon \to u$, $p_\epsilon \to p$ as $\epsilon \to 0$ (in appropriate spaces, up to subsequences), u and p are suitable weak solution to N.S.

EX 3: CONSTRUCTION BY HYPERVISCOSITY

- Add a vanishing hyperviscosity (Lions (1959). Ω is the d-torus, d is the space dimension.

$$\begin{cases} \partial_t u_\epsilon + u_\epsilon \cdot \nabla u_\epsilon + \nabla p_\epsilon - \nu \nabla^2 u_\epsilon + \varepsilon^{2\alpha}(-\nabla^2)^\alpha u_\epsilon = f, \\ \nabla \cdot u_\epsilon = 0 \\ u_\epsilon \text{ is periodic,} \\ u_\epsilon|_{t=0} = u_0. \end{cases}$$

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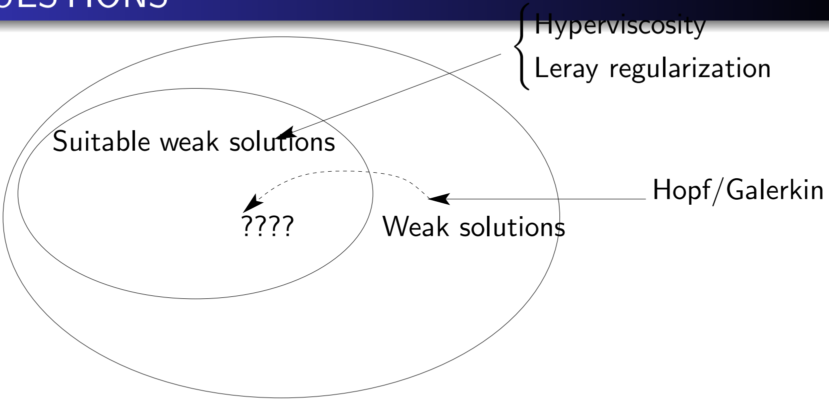
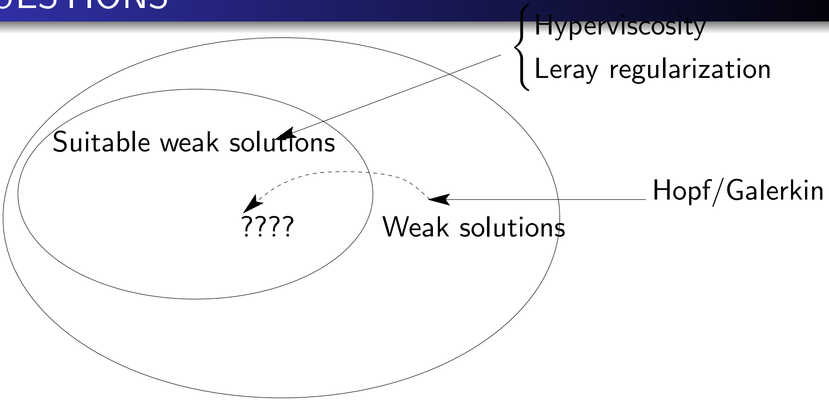
Theorem (Lions (1959), Beirão da Veiga (1985))

Unique weak solution for all $t > 0$ if $\alpha > \frac{d+2}{4}$, and $u_\epsilon \underset{\epsilon \to 0}{\longrightarrow} u$ (up to subsequences) and u is suitable.

QUESTIONS


Hyperviscosity
Leray regularization
Suitable weak solutions
Hopf/Galerkin
????
Weak solutions

QUESTIONS



Q1: Is the set of suitable solutions a proper subset of weak solutions?

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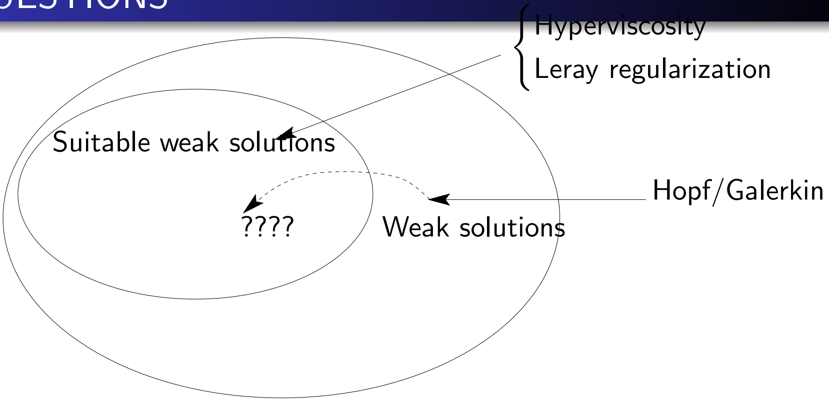

Q1: Is the set of suitable solutions a proper subset of weak solutions?

Q2: Do the Galerkin solutions end up to be suitable after all?

OUTLINE

Boris Grigorievich Galerkin

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- Modify the nonlinear term as follows:

$$b_h(u,v,v) = \begin{cases} (u\cdot\nabla u + \frac{1}{2}u\nabla\cdot u, v) & \text{(Temam, 1967)} \\ ((\nabla\times u)\times u + \frac{1}{2}\nabla(\mathcal{K}_h(u^2)), v) \end{cases}$$

where $\mathcal{K}_h : L^2(\Omega) \longrightarrow M_h$, linear L^2-stable approximation operator.

HYPOTHESES/DEFINITIONS (ctd.)

Definition (Discrete commutator property)

There is an operator $P_h \in \mathcal{L}(H^1_\#(\Omega); X_h)$ (resp. $Q_h \in \mathcal{L}(L^2(\Omega); M_h)$) such that for all ϕ in $W^{2,\infty}_\#(\Omega)$ (resp. all ϕ in $W^{1,\infty}_\#(\Omega)$) and all $v_h \in X_h$ (resp. all $q_h \in M_h$)

$$\|\phi v_h - P_h(\phi v_h)\|_{H^l} \le c\, h^{1+m-l} \|v_h\|_{H^m} \|\phi\|_{W^{m+1,\infty}}, \quad 0 \le l \le m \le 1$$
$$\|\phi q_h - Q_h(\phi q_h)\|_{L^2} \le c\, h \|q_h\|_{L^2} \|\phi\|_{W^{1,\infty}}.$$

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- FE and wavelet-based approximation spaces have the discrete commutator property (local interpolation properties).
- Fourier-based approximation spaces do not have the discrete commutator property (No local interpolation properties).

GALERKIN FORMULATION

- Seek $u_h \in \mathcal{C}^1([0,T];X_h)$ and $p_h \in \mathcal{C}^0([0,T];M_h)$ such that for all $v_h \in X_h$, all $q_h \in M_h$, and all $t \in [0,T]$

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Under the above hypotheses, if X_h and M_h have the discrete commutator property, the couple (u_h, p_h) convergences to a suitable solution to NS.

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- Question was open since Scheffer (1977).

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Question: Does the result hold for Dirichlet BCs?

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Johann Peter Gustav Lejeune Dirichlet

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- Discrete norm $\|v_h\|_{\mathbf{V}_h^s} := (A_h^s v_h, v_h)^{\frac{1}{2}}, \qquad \forall s \in \mathbb{R}.$

PRELIMINARY RESULTS

Lemma

$\exists c_l > 0$ *(non-increasing function)* $\exists c_u > 0$ *(non-decreasing function), independent of* h:

$$c_l(|s|)\|v_h\|_{\widetilde{\mathbf{H}}^s_0} \le \|v_h\|_{\mathbf{V}^s_h} \le c_u(|s|)\|v_h\|_{\widetilde{\mathbf{H}}^s_0}, \qquad \begin{cases} -\frac{1}{2} < s < \frac{3}{2}, & \textit{lower},\\ -\frac{3}{2} < s < \frac{3}{2}, & \textit{upper} \end{cases}$$

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Lemma

$\exists c_l > 0$ (non-increasing function) $\exists c_u > 0$ (non-decreasing function), independent of h:

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and for all $s \in (-\frac{3}{2}, 0]$

$$c_l(|s|)\|\nabla_h^2 v_h\|_{\widetilde{\mathbf{H}}_0^s} \le \|A_h v_h\|_{\mathbf{V}_h^s} \le c_u(|s|)\|\nabla_h^2 v_h\|_{\widetilde{\mathbf{H}}_0^s}, \quad \forall v_h \in \mathbf{V}_h.$$

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$\Rightarrow$ Hopf and Leray solutions are suitable

THE MAIN RESULT (ctd.)

Lemma

There is c independent of h so that,

$$\|\partial_t u_h\|_{H^{\tau-1}((0,T);\mathbf{H}^{-\alpha}(\Omega))} + \|u_h\|_{H^{\tau}((0,T);\mathbf{H}^{-\alpha}(\Omega))} \leq c,$$

for all $\alpha \in [\frac{1}{4}, \frac{1}{2})$ *and for all* $\tau < \overline{\tau} := \frac{2}{5}(1+\alpha)$.

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Lemma

There is c independent of h such that for $s \in [\frac{3}{10}, \frac{1}{2}]$

$$\|p_h\|_{H^{-r}((0,T);H^s(\Omega))} \leq c,$$

for all $r > \overline{r} = \frac{1}{4} + \frac{s}{2}$.

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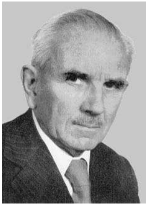

Jean Leray

Heinz Hopf

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- At high *Re* numbers, CFD is always under-resolved. For practical purposes $Re \approx \infty$.
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- Q: Should we bother about suitability?

UNDER-RESOLVED SIMULATIONS

- At high Re numbers, CFD is always under-resolved. For practical purposes $Re \approx \infty$.
- h is never small enough to guaranty suitability of the approximation.
- Q: Should we bother about suitability?
- A: Yes. (What does suitability means after all)?

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$\Rightarrow$ The singular sub-scales (if any) are dissipative (at very small scales energy is dissipated)

A NEW SUBGRID VISCOSITY MODEL?

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- Under-resolved computations $\Leftrightarrow$ There are singular subscales
- To guaranty that at the grid scale h, energy is well dissipated (suitability) we should have

$$R_h(x,t)\cdot u_h \leq 0, \qquad \forall x, t$$

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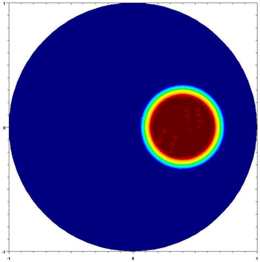
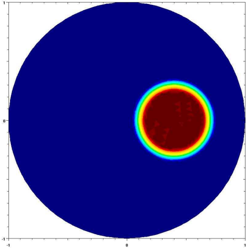
- Note that $|D_h(x,t)| \to 0$ if there is no subgrid scale! (no consistency problem).

OUTLINE



Stabilization? Entropy? Linear transport?

- Solve the transport equation $\partial_t u + \beta \cdot \nabla u = 0$



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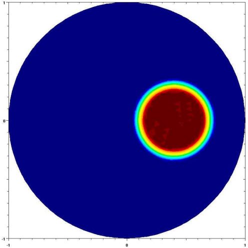

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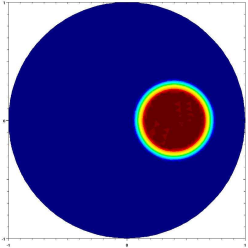

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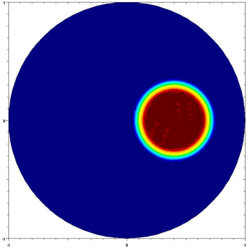

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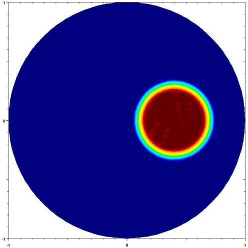

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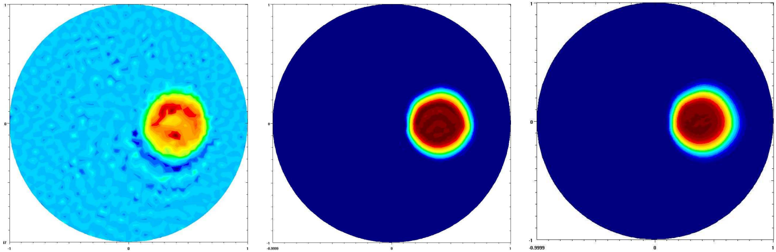
- Convergence $\mathbb{P}_k$: L^1-norm $\mathcal{O}(h^{\frac{k+1}{k+2}})$, L^2-norm $\mathcal{O}(h^{\frac{k+1}{2(k+2)}})$?

Stabilization? Entropy? Linear transport?

- Numerical test $\mathbb{P}_1$, $h = 0.05$, $T = 1$



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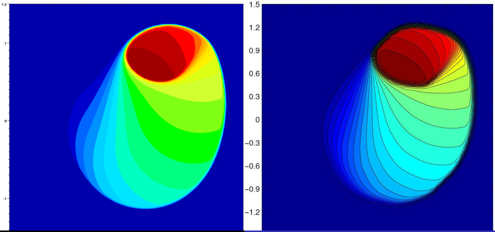

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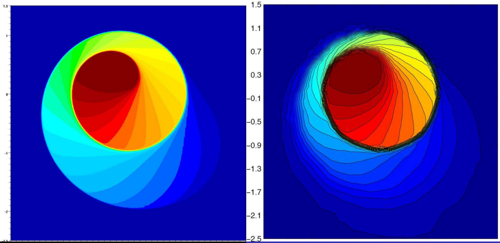

Euler flows

- Solve 1D Euler equations for perfect gas, $(\gamma-1)e = T = p/\rho$, $\gamma = 1.4$
- Entropy $S = \frac{\rho}{\gamma-1}\log(p/\rho^\gamma)$
- Entropy residual, $D_h(u) := \partial_t S + \partial_x(uS)$
- Define wave speed $\beta := |u| + (\gamma T)^{\frac{1}{2}}$

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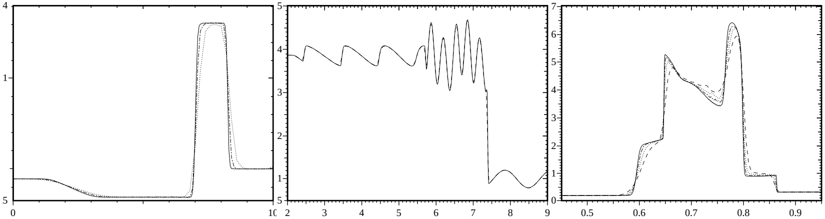

Figure: Lax shock tube, $t = 1.3$, 50, 100, 200 points. Shu-Osher shock tube, $t = 1.8$, 400, 800 points. Right: Woodward-Collela blast wave, $t = 0.038$, 200, 400, 800, 1600 points.

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- A new (very simple) stabilization technique has been proposed and tested.