Minimal stabilization techniques for incompressible flows

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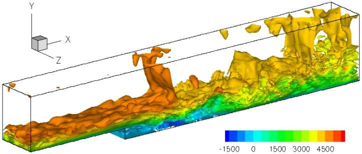

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Outline

F. Brezzi/ M. Fortin: *A minimal stabilisation procedure for mixed finite element methods*, Numer. Math. 89 (2001), 457-492.

- **Goal:** Critical review of stabilization techniques for inf-sup stable pairs (in view of VMS-methods)
- **Acknowledgments:** Thanks to G. Matthies, J. Löwe, T. Heister and X. Zhang.

Treatment of nonstationary laminar Navier-Stokes problem

Find $U = (\mathbf{u}, p) \in V \times Q := \left(H_0^1(\Omega)\right)^d \times L_0^2(\Omega)$ such that

$$B\big(U, V\big) = \big(\tilde{\mathbf{f}}, \mathbf{v}\big) \qquad \forall V = \big(\mathbf{v}, q\big) \in V \times Q \tag{1}$$

$$B\big(U, V\big) := \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \big((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}\big) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}).$$

- Semidiscretise (1) first in time (e.g., BDF(q) or SDIRK methods).
- Newton-type iteration in each time step leads to Oseen type problem:

Find $U = (\mathbf{u}, p) \in V \times Q$ such that

$$a\big(U, V\big) = \big(\mathbf{f}, \mathbf{v}\big) \qquad \forall V = \big(\mathbf{v}, q\big) \in V \times Q.$$

$$a\big(U, V\big) := \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \big((\mathbf{b} \cdot \nabla)\mathbf{u} + \sigma\mathbf{u}, \mathbf{v}\big) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u})$$

with given $\mathbf{b} \in H(\operatorname{div}, \Omega) \cap \left(L^\infty(\Omega)\right)^d$, $\operatorname{div}\mathbf{b} = 0$, $\quad \nu > 0$ and $\frac{1}{\Delta t} \sim \sigma \geq 0$

Galerkin finite element discretization

- $\mathcal{T}_h$ – shape-regular decomposition of polyhedral domain Ω
- $\mathbb{Y}^r_{\mathcal{T}_h} := \{v \in C(\bar{\Omega}) \mid v|_K \in \mathbb{P}_r(K) \text{ or } \mathbb{Q}_r(K) \ \forall K \in \mathcal{T}_h\}, \quad r \in \mathbb{N}$
- FE spaces for velocity/ pressure:

$$\mathbf{V}^r_h := \left[\mathbb{Y}^r_{\mathcal{T}_h} \cap H^1_0(\Omega)\right]^d, \qquad \mathbf{Q}^{r-1}_h := \mathbb{Y}^{r-1}_{\mathcal{T}_h} \cap L^2_0(\Omega)$$

with discrete inf-sup compatibility condition

Galerkin FEM:

$$\text{Find} \quad U = (\mathbf{u}, p) \in \mathbf{W}^{r,r-1}_h := \mathbf{V}^r_h \times \mathbf{Q}^{r-1}_h, \text{ s.t.}$$
$$a(U, V) = (\mathbf{f}, \mathbf{v}) \quad \forall\, V = (\mathbf{v}, q) \in \mathbf{W}^{r,r-1}_h$$

Goal: Robustness w.r.t. ν, σ, h

"Classical" residual-based stabilisation (RBS)

Residual-based scheme:

$$\text{Find } U = (\mathbf{u}, p) \in \mathbf{W}_h^{r,r-1} = \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}, \text{ s.t.}$$
$$a_{rbs}(U,V) = l_{rbs}(V) \quad \forall\, V = (\mathbf{v}, q) \in \mathbf{W}_h^{r,r-1}$$

$$a_{rbs}(U,V) := a(U,V) + \underbrace{\sum_{K\in\mathcal{T}_h} \left(L_{Os}(\mathbf{u},p),\ \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q)\right)_K}_{SUPG-\ and\ PSPG-stabilisation} + \underbrace{\sum_{K\in\mathcal{T}_h} \left(\gamma_K\,(\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}\right)_K}_{(div-)div-stabilisation}$$

$$l_{rbs}(V) := (\mathbf{f},\mathbf{v})_\Omega + \overbrace{\sum_{K\in\mathcal{T}_h} \left(\mathbf{f}, \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q)\right)_K}$$

Other variants:

- Galerkin/ Least-squares method (GaLS): Test with $\tau_K L_{Os}(\mathbf{v}, q)$
- Algebraic subgrid-scale method ("unusual" GaLS): Test with $-\tau_K L_{Os}^*(\mathbf{v}, q)$

Drawbacks of classical RBS-schemes

A-priori analysis: with emphasis on r-dependence

see: GL/ G. Rapin M^3AS 16 (2006) 7

Con's of RBS-schemes

- Basic drawback: Strong velocity-pressure coupling in SUPG-terms !
  $\Rightarrow$ (Rather) expensive implementation in 3D !
- Sensitive design of parameters τ_K and γ_K
- Non-symmetric form of stabilisation terms
- Construction of efficient preconditioners for mixed algebraic problem !

Problem:

Is PSPG-stabilization necessary for div-stable pairs ?

Reduced RBS-scheme

Reduced RBS scheme: Joint work with G. Matthies, L. Röhe (2008)

$$\text{Find } U = (\mathbf{u}, p) \in \mathbf{W}_h^{r,r-1} = \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}, \text{ s.t.}$$

$$a_{red}(U, V) = l_{red}(V) \quad \forall\, V = (\mathbf{v}, q) \in \mathbf{W}_h^{r,r-1}$$

$$a_{red}(U,V) := a(U,V) + \underbrace{\sum_{K \in \mathcal{T}_h} \left(L_{Os}(\mathbf{u},p),\ \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v})\right)_K}_{SUPG-stabilisation} + \underbrace{\sum_{K \in \mathcal{T}_h} \left(\gamma_K\,(\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}\right)_K}_{(div-)div-stabilisation}$$

$$l_{red}(V) := (\mathbf{f}, \mathbf{v})_\Omega + \overbrace{\sum_{K \in \mathcal{T}_h} \left(\mathbf{f}, \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v})\right)_K}$$

with Oseen operator

$$L_{Os}(\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u} + \nabla p$$

Stability analysis

Seminorm/ norm:

$$|[\mathbf{v}]|_{red} := \Big(\nu|\mathbf{v}|_1^2 + \sigma\|\mathbf{v}\|_0^2 + \gamma\|\nabla\cdot\mathbf{v}\|_0^2 + \sum_K \tau_K\|\mathbf{b}\cdot\nabla\mathbf{v}\|_{0,K}^2\Big)^{\frac{1}{2}}$$

$$|||V|||_{red} := \Big(|[\mathbf{v}]|_{red}^2 + \alpha\|q\|_0^2\Big)^{\frac{1}{2}}$$

Conditional stability:

- $\gamma_K \equiv \gamma \geq 0, \qquad 0 \leq \tau_K \leq \frac{\beta_0^2}{30\mu^2}\,\frac{\mathbf{h_K^2}}{\varphi^2}$

  with $\varphi^2 := \nu + \sigma C_F^2 + \|\mathbf{b}\|_{L^\infty(\Omega)}^2 \min\left(\frac{1}{\sigma}; \frac{C_F^2}{\nu}\right) + \gamma$

- Set: $\frac{16}{15}\cdot\frac{\beta_0^2}{\varphi^2} \leq \alpha \leq \frac{26}{15}\cdot\frac{\beta_0^2}{\varphi^2}$

$\rightsquigarrow$

$$\exists\ \beta_S \neq \beta_S(\nu,\sigma,h) : \qquad \inf_{V_h}\sup_{W_h} \frac{a_{red}(V_h, W_h)}{|||V_h|||_{red}\ |||W_h|||_{red}} \geq \beta_S > 0$$

Convergence result. Parameter design

Preliminary a-priori estimate:

- Let $(\mathbf{u}, p) \in [\mathbf{V} \cap H^{r+1}(\Omega)]^d \times [\mathbf{Q} \cap H^r(\Omega)]$.
- Stability assumption implies $\tau_K \leq \frac{Ch_K^2}{\gamma+\nu+\sigma C_F^2}$.

$$|||U - U_h|||^2_{red} \leq C \sum_K \Big(\frac{h_K^2}{\nu + \gamma} h_K^{2(r-1)} \|p\|^2_{r,K} + \Big[\gamma + \nu + \sigma h_K^2 + \tau_K \|\mathbf{b}\|^2_{\infty,K} + \frac{\|\mathbf{b}\|^2_{\infty,K} h_K^2}{\tau_K \|\mathbf{b}\|^2_{\infty,K} + \nu + \sigma h_K^2}\Big] h_K^{2r} \|\mathbf{u}\|^2_{r+1,\omega(K)} \Big)$$

Conclusions for stabilization:

- SUPG **not** required if: $\nu \geq \|\mathbf{b}\|^2_{\infty,K} h_K^2$, i.e. $\max_K Re_K \leq \frac{1}{\sqrt{\nu}}$
  and/ or $\sigma \geq \|\mathbf{b}\|^2_{\infty,K}$ $\rightsquigarrow$ time step restriction: $h_K^2 \lesssim \delta t \sim \frac{1}{\sigma} \lesssim \frac{1}{\|\mathbf{b}\|^2_\infty}$
- Div-stabilization useful if: $\|p\|_{r,K} \sim (\nu + \gamma) \|\mathbf{u}\|_{r+1,\omega(K)}$

Upper bound of critical Galerkin terms

Upper bound of $a_{red}(\cdot,\cdot)$ requires sharp estimates of Galerkin terms:

- Discrete-divergence preserving interpolant I_h GIRAULT/SCOTT ['03]
- Standard Lagrangian interpolant J_h

$\rightsquigarrow$ For all $W_h = (\mathbf{w}_h, r_h) \in \mathbf{V}_h \times \mathbf{Q}_h$:

$$(r_h, \nabla\cdot(\mathbf{u} - I_h\mathbf{u})) = 0 \qquad \text{avoids negative power of pressure weight } \alpha$$

$$|(p - J_h p, \nabla\cdot\mathbf{w}_h)| \leq C\Big(\sum_K \frac{2}{\nu+\gamma} h_K^{2r}\|p\|_{r,K}^2\Big)^{\frac{1}{2}} |||W_h|||_{red}$$

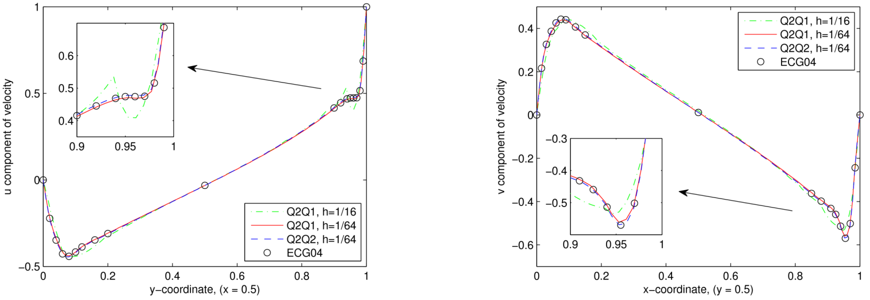
$$\big|(\mathbf{b}\cdot\nabla(\mathbf{u} - I_h\mathbf{u}), \mathbf{w_h})\big| \leq C\Big(\sum_K \frac{3\|\mathbf{b}\|_{\infty,K}^2 h_K^2}{\tau_K\|\mathbf{b}\|_{\infty,K}^2 + \nu + \sigma h_K^2} h_K^{2r}\|\mathbf{u}\|_{r+1,\omega(K)}^2\Big)^{\frac{1}{2}} |||W_h|||_{red}$$

Action of div- resp. SUPG-stabilization avoid negative powers of ν resp. ν, σ

Can SUPG be avoided for laminar flows ?

Example: Driven cavity with stationary solutions

- SUPG is not required up to $Re = 7.500$
- Non-stationary approach with moderately large time steps

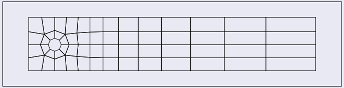


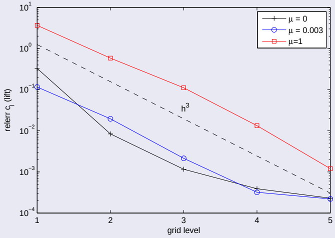
Driven-cavity problem with $Re = 5,000$: Cross-sections of the solutions for Q_2/Q_1 without SUPG/PSPG and Q_2/Q_2 with SUPG/PSPG

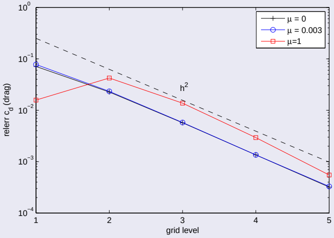
Role of grad-div stabilization I

Examples with $\|p\|_{r,K} \ll \|\mathbf{u}\|_{r+1,\omega(K)}$

- Poiseuille flow: $\nabla p = \nu \Delta u$ $\rightsquigarrow$ div-stabilization superfluous
- Stationary flow around cylinder at $\nu = 0.001$ (corresponds to $Re = 20$)







Convergence plots with Q_2/Q_1 for lift and drag coefficients

Role of grad-div stabilization II

Examples with $\|p\|_{r,K} \sim \|\mathbf{u}\|_{r+1,\omega(K)} \quad \rightsquigarrow \gamma \sim 1 \gg \nu$

$$-\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}$$

with solution $\mathbf{u} = \mathbf{b} = (\sin(\pi x_1), -\pi x_2 \cos(\pi x_1))^T, \;\; p = \sin(\pi x_1)\cos(\pi x_2)$

Table: Comparison of different variants of stabilization with Q_2/Q_1 and $\nu = 10^{-6}, \sigma = 1, h = \frac{1}{64}$

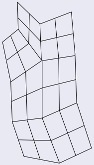
SUPG: τ_0	div: γ_0	PSPG: α_0	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_1$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\|\nabla\cdot\mathbf{u}_h\|_0$	$\|p-p_h\|_0$
0.000	0.000	0.000	2.56E-1	5.42E-4	2.02E-1	2.31E-4
0.056	0.562	0.010	1.91E-3	6.21E-6	1.82E-4	9.08E-5
0.056	0.562	0.000	1.91E-3	6.20E-6	1.66E-4	8.06E-5
0.000	0.562	0.000	2.61E-3	7.42E-6	1.72E-4	8.05E-5
3.162	0.000	0.000	1.87E-2	7.50E-5	1.56E-2	1.08E-4

Problem:

General criterion for div-div stabilization *or* a-posteriori approach ?!

Two-grid setting and local projection

- **Primal grid** $\mathcal{T}_h$ with FE spaces $\mathbf{V}_h^r$ and $\mathbf{Q}_h^{r-1}$ for velocity and pressure



- **Macro grid** $\mathcal{M}_h = \mathcal{T}_{2h}$ with discontinuous FE spaces
  - $\mathbf{D}_h^u := \{v \in [L^2(\Omega)]^d : v|_M \in \mathbb{Y}_{\mathcal{M}_h}^{r-1}, \forall M \in \mathcal{M}_h \}$
  - $\mathbf{D}_h^p := \{v \in L^2(\Omega) : v|_M \in \mathbb{Y}_{\mathcal{M}_h}^{k-1}, \forall M \in \mathcal{M}_h \}$, $k \in \{0, \ldots, r-2\}$

- **Local L^2-projection**: $\pi_M^{u/p} : L^2(M) \to \mathbf{D}_h^{u/p}|_M$
- Global projection: $\pi_h^{u/p} : L^2(\Omega) \to \mathbf{D}_h^{u/p}$, $(\pi_h^{u/p} w)|_M := \pi_M^{u/p}(w|_M)$

Local projection stabilisation

Fluctuation operators:

- $\kappa_h^{u/p} : [L^2(\Omega)] \to [L^2(\Omega)], \qquad \kappa_h^{u/p} := id - \pi_h^{u/p}$
- $\vec{\kappa}_h^{u/p} : [L^2(\Omega)]^d \to [L^2(\Omega)]^d, \qquad \vec{\kappa}_h^{u/p}\vec{w} := \left((id - \pi_h^{u/p})w_i\right)_{i=1}^d$

Discrete LPS-problem: Subgrid stabilization as minimal stabilization

Find $U_h = (\mathbf{u}_h, p_h) \in \mathbf{W}_h^{r,r-1} : \quad a_{lps}(U_h, V) = (\mathbf{f}, \mathbf{v}) \ \forall \, V = (\mathbf{v}, q) \in \mathbf{W}_h^{r,r-1}$

$$a_{lps}(U, V) = a(U, V) + s_h(U, V).$$

$$s_h(U,V) = \sum_M \underbrace{\alpha_M(\vec{\kappa}_h^u \nabla p, \vec{\kappa}_h^u \nabla q)_M}_{pressure \quad stab.} + \underbrace{\tau_M\left(\vec{\kappa}_h^u \mathbf{b} \cdot \nabla \mathbf{u}, \vec{\kappa}_h^u \mathbf{b} \cdot \nabla \mathbf{v}\right)_M}_{advection \quad stab.} + \underbrace{\gamma_M(\kappa_h^p \nabla \cdot \mathbf{u}, \kappa_h^p \nabla \cdot \mathbf{v})_M}_{divergence \quad stab.}$$

Stability

Comparison of LPS to RBS-schemes:

- Symmetric, non-consistent form of stabilization terms
- Stabilization (or: subgrid viscosity) term acts only on "fine" scales (!)

$$\begin{aligned} |||V_h|||_{lps} &:= \left(|[V_h]|^2_{lps} + \alpha\|q\|^2_0\right)^{\frac{1}{2}}, \qquad \alpha = \alpha(\nu,\sigma) > 0 \\ |[V_h]|_{lps} &:= \left(\nu\|\nabla \mathbf{v}_h\|^2_0 + \sigma\|\mathbf{v}_h\|^2_0 + s_h(V_h,V_h)\right)^{\frac{1}{2}} \end{aligned}$$

Unconditional (!) stability: $\Longrightarrow$ Existence / uniqueness

$$\inf_{V_h\in\mathbf{W}_h^{r,r-1}} \sup_{W_h\in\mathbf{W}_h^{r,r-1}} \frac{(a+s_h)(V_h,W_h)}{|[V_h]|_{lps}\ |[W_h]|_{lps}} \geq 1$$

$$\exists\beta_S \neq \beta_S(\nu,h): \quad \inf_{V_h\in\mathbf{W}_h^{r,r-1}} \sup_{W_h\in\mathbf{W}_h^{r,r-1}} \frac{(a+s_h)(V_h,W_h)}{|||V_h|||_{lps}\ |||W_h|||_{lps}} \geq \beta_S > 0$$

A-priori error estimate

Technical ingredient: see talk of G. Matthies

Construction of special interpolation operator $\vec{j}_h^u$

- s.t. $\mathbf{v} - \vec{j}_h^u \mathbf{v}$ is L^2-orthogonal to $\mathbf{D}_h^u$ for all $\mathbf{v} \in \mathbf{V}$
- which preserves the discrete divergence constraint.

Preliminary a-priori estimate:

- *Let* $\mathbf{u} \in [H_0^1(\Omega) \cap H^{r+1}(\Omega)]^d, \;\; p \in L_0^2(\Omega) \cap H^r(\Omega).$

$\Rightarrow \;\; \exists C \neq C(\nu, \sigma, h)$:

$$|[U - U_h]|_{lps}^2 \;\leq\; C \sum_{M \in \mathcal{M}_h} \Big(\big(\alpha_M + \frac{h_M^2}{\gamma_M}\big) h_M^{2(r-1)} \|p\|_{r,\omega_M}^2 + \tau_M h_M^{2r} \|\mathbf{b} \cdot \nabla \mathbf{u}\|_{r,\omega_M}^2$$
$$+ \;\; \big(\nu + \sigma h_M^2 + \gamma_M + \frac{h_M^2}{\tau_M} + \|\mathbf{b}\|_{\infty,M}^2 \tau_M\big)\, h_M^{2r} \|\mathbf{u}\|_{r+1,\omega_M}^2 \Big)$$

Parameter design

Conclusions for parameter design:

- **PSPG-type term:** $\alpha_M = 0$ is possible !

- **SUPG-type term:** $\tau_M \sim \frac{h_M}{\|\mathbf{b}\|_{\infty,M}}$

  Careful analysis $\rightsquigarrow$ SUPG avoidable if $\sigma \geq \|\mathbf{b}\|_\infty^2$ or $\max_M Re_M \leq \frac{1}{\sqrt{\nu}}$

- **Div-type term:**

  Equilibration of red velocity and pressure terms with $\alpha_M = 0$ yields

$$\gamma_M \|\mathbf{u}\|_{r+1,\omega_M} \sim \|p\|_{r,\omega_M}$$

$\rightsquigarrow$ Same problem (and strategies) as for reduced RBS-scheme !

VMS-decomposition of Navier-Stokes problem

Navier-Stokes problem :

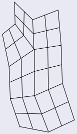
Find $U = (\mathbf{u}, p) \in \mathcal{V}$ s.t. $\mathbf{u}(0) = \mathbf{u}_0$ and

$$B(U,V) = \langle \mathbf{f}, v \rangle \qquad \forall V = (\mathbf{v}, q) \in \mathcal{W}$$

Decomposition of trial and test spaces:

Two-level setting with FE spaces:

$\mathcal{V}_H \subseteq \mathcal{V}_h \subset \mathcal{V}, \quad \mathcal{W}_H \subseteq \mathcal{W}_h \subset \mathcal{W}$



$$\mathcal{V} = \underbrace{\mathcal{V}_H \oplus (id - \Pi)\mathcal{V}_h}_{=:\mathcal{V}_h} \oplus \hat{\mathcal{V}} \quad \rightsquigarrow \quad U = \underbrace{U_H + (id - \Pi)U_h}_{=:U_h} + \hat{U}$$

$$\mathcal{W} = \underbrace{\mathcal{W}_H \oplus (id - \Pi)\mathcal{W}_h}_{=:\mathcal{V}_h} \oplus \hat{\mathcal{W}} \quad \rightsquigarrow \quad V = \underbrace{V_H + (id - \Pi)V_h}_{=:U_h} + \hat{V}$$

Discrete VMS problem on FEM-level

VMS assumptions:

A.1 Scale separation: No direct influence of $\hat{U}$ on U_H

A.2 Unresolved scales dissipate energy from small resolved scales via **subgrid viscosity model** $S: \; \mathcal{V}_h \cup \mathcal{W}_h$

Discrete VMS problem:

$$\begin{aligned} B(U_h, V_H) &= \langle \mathbf{f}, \mathbf{v}_H \rangle && \forall V_H \in \mathcal{V}_H \\ B(U_h, \tilde{V}_h) + S_h(\tilde{U}_h, \tilde{V}_h) &= \langle \mathbf{f}, \tilde{\mathbf{v}}_h \rangle && \forall \tilde{V}_h \in (id - \Pi)\mathcal{W}_h \end{aligned}$$

Assumption: $S_h(\cdot, V_H) = 0 \quad \forall V_H \in \mathcal{V}_H \cup \mathcal{W}_H \quad \rightsquigarrow$

Compact discrete VMS problem:

Find $U_h \in \mathcal{V}_h : \; B(U_h, V) + S_h(U_h, V) = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall V \in \mathcal{W}_h$

Simplest parametrization of subgrid models: MILES

Assumption on subgrid viscosity model:

(i) Symmetry: $S_h(U,V) = S_h(V,U) \quad \forall U,V \in \mathcal{V}_h \cup \mathcal{W}_h$

(ii) Coercivity: $S_h(\tilde{U},\tilde{U}) \geq c\|\nabla \tilde{U}\|^2 \quad \forall \tilde{U} \in \tilde{\mathcal{V}} := (id - \Pi)\mathcal{V}_h$

Variational multiscale method (VMS):

- **Scale separation:** Π as L^2-orthogonal projection of $\mathcal{V}_h$ onto $\mathcal{V}_H$
- **Subgrid viscosity model:** $\rightsquigarrow$ local projection stabilization (LPS)

$$S_h(U,V) = \sum_{M \in \mathcal{T}_H} \Big(\tau_M^H D((id-\Pi)\mathbf{u}), D((id-\Pi)\mathbf{v}) \Big)_M$$

- **Parametrization of τ_M^H:** based on a-priori analysis for linearized models

$$\tau_M^H = \tau_0 \|\mathbf{u}\|_{L^\infty(M)} h_M, \qquad \tau_0 = ?$$

corresponds to **m**onotonically **i**ntegrated **LES** (MILES)

Simplest parametrization of turbulence subgrid model

Smagorinsky-type VMS-method: V. John et al. 2008

- **Scale separation:** Π as L^2-orthogonal projection of $\mathcal{V}_h$ onto $\mathcal{V}_H$
- **Subgrid viscosity model:**

$$S_h(U,V) = \sum_{M\in\mathcal{T}_H} \Big(\tau_M^H(\mathbf{u}) D((id-\Pi)\mathbf{u}), D((id-\Pi)\mathbf{v})\Big)_M$$

$$\tau_M^H(\mathbf{u}) = (C_S\, h_M)^2 \|D(\mathbf{u})\|_F, \qquad C_S = ?$$

Potential compromise: see also talk by J. L. Guermond

$$S_h(U,V) = \sum_{M\in\mathcal{T}_H} \Big(\tau_M^H(\mathbf{u}) D((id-\Pi)\mathbf{u}), D((id-\Pi)\mathbf{v})\Big)_M$$

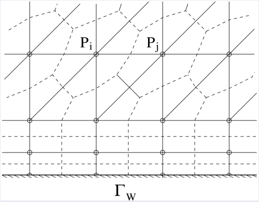
$$\tau_M^H(\mathbf{u}) = \min\Big(\tau_0 \|\mathbf{u}\|_{L^\infty(M)} h_M; (C_S\, h_M)^2 \|D(\mathbf{u})\|_F\Big)$$

Experience with low-order finite-volume code

Finite-volume code Theta (DLR Göttingen):

- Vertex-based variant $\sim$ P_1/P_1 or Q_1/Q_1

  $\rightsquigarrow$ pressure stabilization required
- Chorin decoupling of velocity / pressure



Minimal stabilization:

- Galerkin discretization as upwind stabilization "dissipates" fluctuations
- Subgrid viscosity model: Smagorinsky model

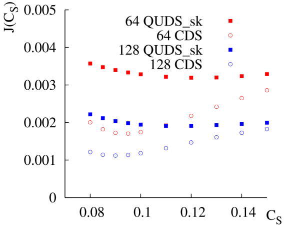
$$S_h(U,V) = \big((C_S \Delta)^2 \|D(\mathbf{u})\|_F D(\mathbf{u}), D(\mathbf{v}) \big), \qquad C_S = ?$$

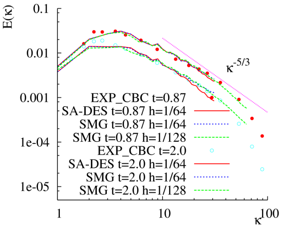
- Experiments by X. Zhang (2007, 08)

Decaying homogeneous turbulence (DHT)

Basic calibration model: Decaying homogeneous turbulence

- Results for turbulent kinetic energy $k = \frac{1}{2}\langle(u - \langle u\rangle)^2\rangle$
- Careful experimental data for k (Comte/Bellot)
- Fourier space characterization of $k(t)$ via energy spectral density $E(\kappa, t)$
- Cost functional: $J(C_S) = \frac{1}{2}\sum_{j=1}^{2}\sum_{i=1}^{M}\left[E(\kappa_i, C_S, t_j) - E_{\exp}(\kappa_i, t_j)\right]^2$

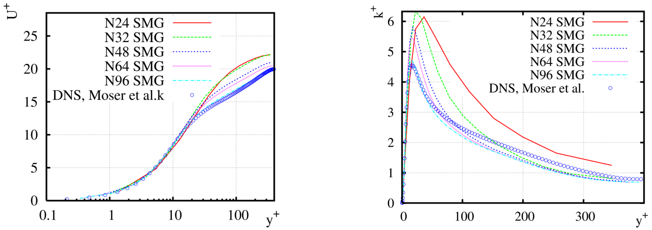




Calibration of Smagorinsky constant C_S and "optimized" energy spectrum

Turbulent channel flow at $Re_\tau = 395$

- Anisotropic resolution of boundary layer region
- First-order statistics: mean streamwise velocity $U = \langle u \rangle e_1$
- Second-order statistics: turbulent kinetic energy $k = \frac{1}{2}\langle (u - \langle u \rangle)^2 \rangle$
- and their normalized variants $U^+ = U/u_\tau, \quad k^+ = k/u_\tau^2$



Grid convergence with N^3 nodes

Problem: Wall-resolved LES is almost as expensive as DNS

Proper resolution of near-wall region in LES requires $\sim \mathrm{Re}_\tau^2$ nodes (Bagett et al. 1997)

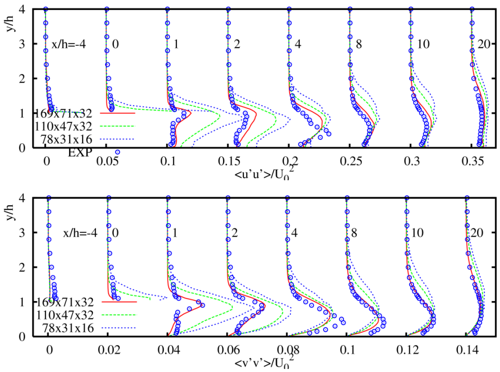
DES for backward facing step at $Re_h = 37.500$

Hybrid approach:

– LES simulation away from boundary layers

– RANS type universal wall-functions used to bridge near-wall region ($y^+ \lesssim 40$)

⇝ significant savings in grid points and allows to increase time step



Summary. Outlook

Summary

- PSPG-type stabilization avoidable for div-stable velocity-pressure pairs
- SUPG-type stabilization less important for laminar flows
- New characterization of div-div stabilization
- LPS-type approach as minimal stabilization in LES/DES

Outlook

- FEM with LPS-type LES: MILES vs. turbulent subgrid model
- Model reduction with DES

THANKS FOR YOUR ATTENTION !