FEM Multigrid Techniques for Viscoelastic Flow

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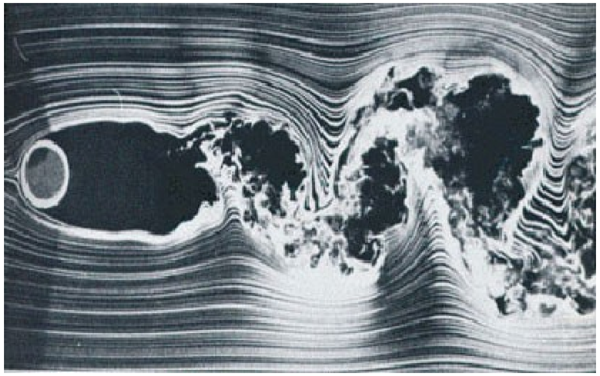
Multiscale CFD Problems

Inertia turbulence

⇨ Re>>1

⇨ Numerical instabilities + problems

Turbulence Models
Stabilization Techniques



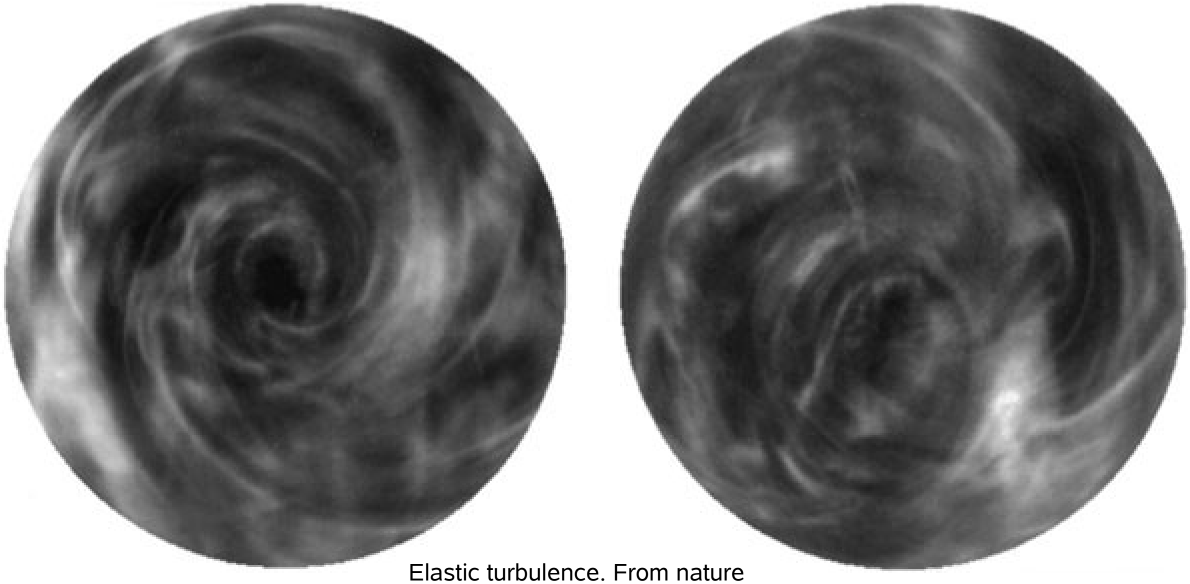
Turbulence flow inside a pipe. From ProPipe

Characteristics:

- Irregular temporal behaviour and spatially disordered
- Broad range of spatial/temporal scales

Multiscale CFD Problems

…looks like developed turbulence…



Elastic turbulence. From nature

Elastic turbulence

⇒ Re<<1, Wi>>1 (less inertia, more elasticity)

⇒ Coil stretching, high stresses

⇒ Numerical instabilities + problems

Flow models: Oldroyd, Maxwell,…
Stabilization: EEME, EEVS, DEVSS/DG, SD, SUPG,…

Nonlinear Flow Models

Generalized Navier-Stokes equations

$$\rho \frac{\partial \mathrm{u}}{\partial t} + \mathrm{u} \cdot \nabla \mathrm{u} - \operatorname{div} \sigma + \nabla p = \rho \mathrm{f}, \ \operatorname{div} \mathrm{u} = 0,$$

$$\frac{\partial \Theta}{\partial t} + \mathrm{u} \cdot \nabla \Theta - \operatorname{div} k \nabla \Theta - \mathrm{D} : \sigma = 0,$$

$$\sigma = \sigma^s + \sigma^p, \ \mathrm{D} = \frac{1}{2}\left(\nabla \mathrm{u} + \nabla \mathrm{u}^T\right)$$

Quasi-Newtonian part $\sigma^s = 2\eta_s\left(\mathrm{D}, \Theta\right)\mathrm{D}$

Viscoleastic part $\sigma^p + \Lambda \frac{\delta_a \sigma^p}{\delta t} = 2\eta_p \mathrm{D},$

$$\frac{\delta_a \sigma}{\delta t} = \left(\frac{\partial}{\partial t} + \mathrm{u} \cdot \nabla\right)\sigma + \frac{1-a}{2}\left(\sigma \nabla \mathrm{u} + \nabla \mathrm{u}^T \sigma\right) - \frac{1+a}{2}\left(\nabla \mathrm{u}\, \sigma + \sigma \nabla \mathrm{u}^T\right)$$

Required: I) Special Models

$$T + \Lambda \frac{\delta_a T}{\delta t} = 2\eta_0 \left(D + \Lambda_r \frac{\delta_a D}{\delta t} \right)$$

Oldroyd A

Oldroyd B

Maxwell A

Maxwell B

Jeffreys

$$T + \Lambda \frac{\delta_a T}{\delta t} + B(T) = 2\eta D$$

Phan-Thien Tanner

Phan-Thien

Giesekus

Required: II) Special Numerics

Special FEM Techniques

Multigrid Solvers

Stabilization for high Re and Wi Numbers

Implicit Approaches

Space-Time Adaptivity

Grid Deformation Methods

Newton Methods

Our Numerical Approach

Fully implicit monolithic multigrid FEM solver

Numerical Techniques

- The FEM techniques have to handle the following challenging points
  - Stable FE spaces for velocity and pressure fields, and velocity and extra-stress fields → Q2/P1/? or Q1(nc)/P0/? (new: Q2(nc)/P1/?)
  - Special treatment of the convective terms $\mathbf{u}\cdot\nabla\mathbf{u}$,$\mathbf{u}\cdot\nabla\Theta$ $\mathbf{u}\cdot\nabla\sigma$

    → edge-oriented/interior penalty FEM, TVD/FCT
  - The presence of the „reactive" terms which are responsible for
    - high Weissenberg number problem (**HWNP**) → LCR
    - blow up phenomena for time dependent solution

- The (nonlinear) solvers have to deal with different source of nonlinearity
  - nonlinear viscosities → Newton method via divided differences
  - the strong coupling of equations → monolithic multigrid approach
  - complex geometries and meshes

Newton Solver

Solve for the residual of the nonlinear system algebraic equations

$$R(\mathrm{x}) = 0, \quad \mathrm{x} = (\mathrm{u}, \Theta, \sigma, p)$$

Newton method with damping results in iterations of the form

$$\mathrm{x}^{n+1} = \mathrm{x}^{n} + \omega^{n} \left[\frac{\partial R(\mathrm{x}^{n})}{\partial \mathrm{x}} \right]^{-1} R(\mathrm{x}^{n})$$

- Continuous Newton: on variational level (before discretization)

→ The continuous Frechet operator can be calculated

- Inexact Newton: on matrix level (after discretization)

→ The Jacobian matrix is **approximated** using finite differences as

$$\left[\frac{\partial R(\mathrm{x}^{n})}{\partial x} \right]_{ij} \approx \frac{R_i(\mathrm{x}^{n} + \varepsilon \mathrm{e}_j) - R_i(\mathrm{x}^{n} - \varepsilon \mathrm{e}_j)}{2\varepsilon}$$

Multigrid Solver

- Standard geometric multigrid approach
- Full Q_2, $\tilde{Q}_1$, P_1^{disc} and P_0 grid transfer
- Smoother: Local/Global MPSC
  - Local MPSC via Vanka-like smoother

$$\begin{bmatrix} \mathrm{u}^{l+1} \\ \sigma^{l+1} \\ \Theta^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \mathrm{u}^{l} \\ \sigma^{l} \\ \Theta^{l} \\ p^{l} \end{bmatrix} + \omega^l \, \Sigma_{T\in\tau_h} \left[K+J\right]^{-1}_{|T} \begin{bmatrix} \mathrm{Res}_{\mathrm{u}} \\ \mathrm{Res}_{\sigma} \\ \mathrm{Res}_{\Theta} \\ \mathrm{Res}_{p} \end{bmatrix}_{|T}$$

Coupled multigrid solver

  - Global MPSC
    - solve for an intermediate $\tilde **u** (generalized momentum equation)
    - solve for *p* (pressure Poisson equation)
    - update of **u** and *p*
    - solve for Θ (tracer equation)
    - solve for σ (constitutive equation)

Decoupled multigrid solver

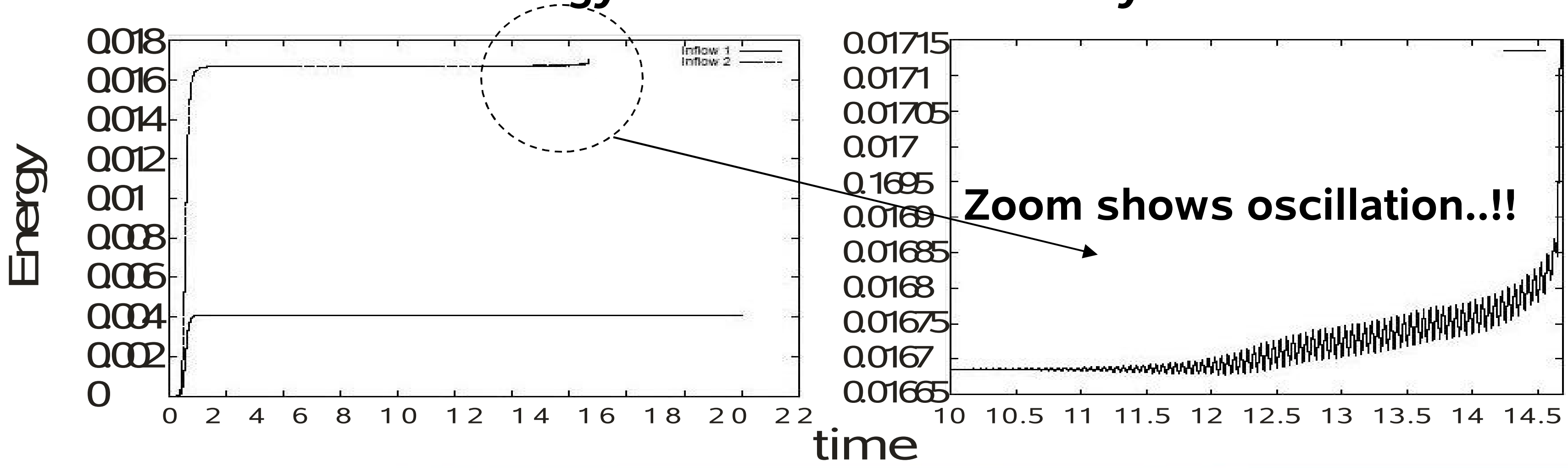
Viscoelastic Models

Different highly developed models

Oldroyd A/B, Maxwell A/B, Jeffreys, PTT, Giesekus

…nevertheless, despite „good“ models and „good“ Numerics, the HWNP („High Weissenberg Number problem“) stills exists for critical Wi, resp., De numbers…

Kinetic Energy for two different velocity inflow



Problem Reformulation

Old → (u, p, $\boldsymbol{\sigma}^p$)

$$\left.\begin{aligned} &\rho\left(\frac{\partial \mathrm{u}}{\partial t}+\mathrm{u}\cdot\nabla\mathrm{u}\right)=\nabla p-2\eta_s\nabla\cdot D-\nabla\cdot\sigma^p,\\ &\nabla\cdot\mathrm{u}=0,\\ &\Lambda\frac{\delta_{\mathrm{a}}\sigma^p}{\delta t}+\sigma^p-2\eta_p D=0, \end{aligned}\right\}\quad (1)$$

Conformation tensor → (u, p, $\boldsymbol{\tau}$)   This tensor is positive definite by design !!

Replace $\boldsymbol{\sigma}^p$ in (1) with $\sigma^p=\frac{\eta_p}{\Lambda}(\tau-\mathrm{I})$ → special discretization : TVD

$$\left.\begin{aligned} &\rho\left(\frac{\partial \mathrm{u}}{\partial t}+\mathrm{u}\cdot\nabla\mathrm{u}\right)=\nabla p-2\eta_s\nabla\cdot D-\frac{\eta_p}{\Lambda}\nabla\cdot\tau,\\ &\nabla\cdot\mathrm{u}=0,\\ &\frac{\delta_{\mathrm{a}}\tau}{\delta t}+\frac{1}{\Lambda}(\tau-\mathrm{I})=0, \end{aligned}\right\}\quad (2)$$

Properties of Conformation Tensor

$$\tau(X,t) = \int_{\infty}^{t} \frac{\eta_p}{\Lambda} \exp\left(\frac{-(t-s)}{\sqrt{\Lambda}} \right) F(s,t)\, F(s,t)^T \, ds$$

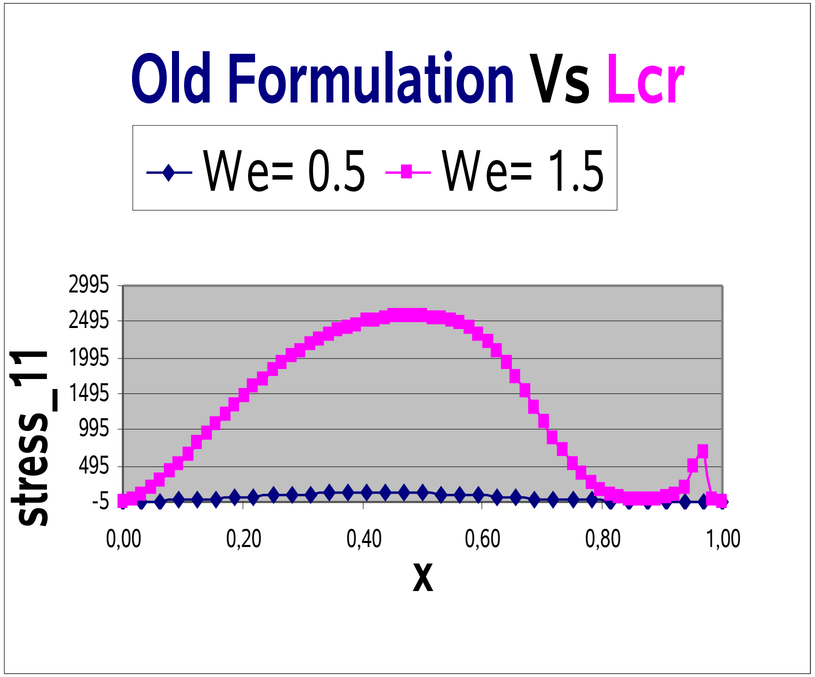
Positive by design,
so we can take its logarithm

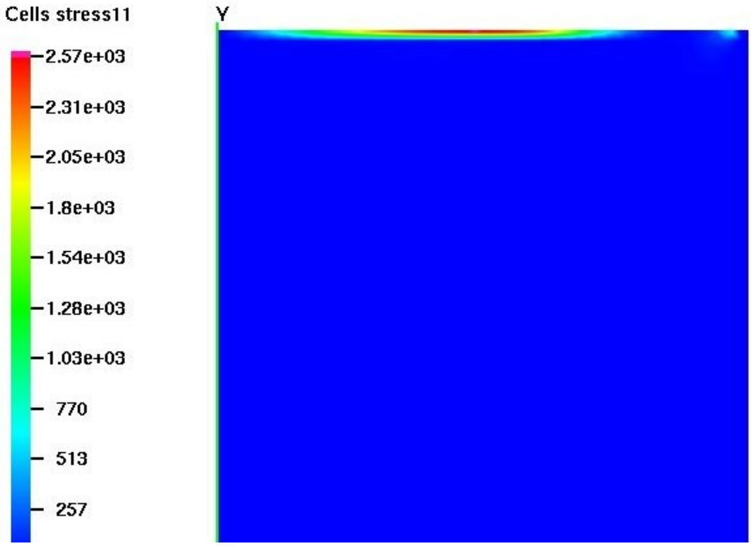
2 observations:

- positive definite → special discretizations like FCT/TVD
- exponential behaviour → approximation by polynomials???

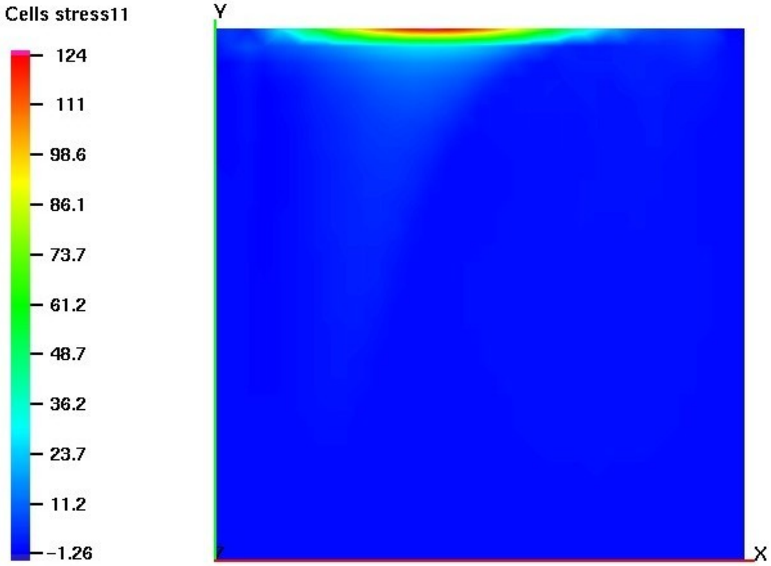
Driven Cavity

Cutline of Stress_11 component at y = 1.0





(LCR)



(Old)

Problem Reformulation

M. Behr → (u, p, ψ)

Replace $\boldsymbol{\tau}$ in (2) with $\tau = \exp\psi$

$$\left.\begin{aligned} &\rho\left(\frac{\partial \mathrm{u}}{\partial t} + \mathrm{u}\cdot\nabla\mathrm{u}\right) = \nabla p - 2\eta_s \nabla\cdot D - \frac{\eta_p}{\Lambda}\nabla\cdot(\exp\psi), \\ &\nabla\cdot\mathrm{u} = 0, \\ &\frac{\delta_{\mathrm{a}}(\exp\psi)}{\delta t} + \frac{1}{\Lambda}\left(\exp\psi - \mathrm{I}\right) = 0, \end{aligned}\right\} \quad (3)$$

Gradient of exponential of ψ → ???

Solvers → ???

LCR Formulation (I)

Experiences:

- Stresses grow exponentially
- Conformation stress is positive by design
- Stretching part creates numerical problem $\left(\nabla \mathrm{u}\, \sigma + \sigma\, \nabla \mathrm{u}^T\right)$

Idea („Kupferman Trick“):

•Decompose the velocity gradient inside the stretching part

$$\nabla u = \Omega + B + N \sigma_c^{-1}$$

•Take the logarithm as a new variable (ψ=log σ) using eigenvalue problem

$$\psi = R \log(\lambda_\tau)\, R^T$$

LCR Formulation (II) $\tau = \exp\psi$

Inside the constitutive equation (2), decompose $\nabla u = \Omega + B + N\sigma_c^{-1}$

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\tau - \left(\Omega\tau - \tau\Omega\right) + 2B\tau = \frac{1}{\Lambda}\left(\mathrm{I} - \tau\right) \qquad (4)$$

Matrix B is purely extension: Responsible for the stretching

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

... it is commutable !!! $B\tau + \tau B = 2B\tau$

Thus... $\frac{\partial \tau}{\partial t} = 2B\tau \Rightarrow \frac{\partial \psi}{\partial t} = 2B$

Matrix Ω is purely rotation: Responsible for the rotating

$$\begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}$$

... it is symmetric !!! $(\Omega\tau - \tau\Omega)_{ij} = (\Omega\tau - \tau\Omega)_{ji}$

Thus... $\frac{\partial \tau}{\partial t} = (\Omega\tau - \tau\Omega) \Rightarrow \frac{\partial \psi}{\partial t} = (\Omega\psi - \psi\Omega)$

LCR Formulation (III)

$$\frac{\partial(\exp\psi)}{\partial t} = \frac{\partial \psi}{\partial t}\exp\psi$$

(u, p, $\boldsymbol{\psi}$)

As in M. Behr, replace in (4) $\boldsymbol{\tau} = \exp \boldsymbol{\psi}$ decouples 2Bτ

$$\left.\begin{aligned} &\rho\left(\frac{\partial \mathrm{u}}{\partial t} + \mathrm{u}\cdot\nabla\mathrm{u}\right) = \nabla p - 2\eta_s \nabla\cdot D - \frac{\eta_p}{\Lambda}\nabla\cdot\exp\psi, \\ &\nabla\cdot\mathrm{u} = 0, \\ &\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\psi - \left(\Omega\psi - \psi\Omega\right) + 2B = \frac{1}{\Lambda}\left(\mathrm{e}^{-\psi} - \mathrm{I}\right), \end{aligned}\right\} \quad (5)$$

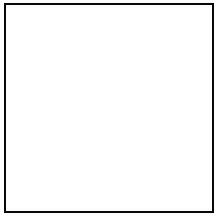
Note: Divergence of exponential of ψ is calculated explicitely using eigenvalue problem !!

Standard discretization techniques →EO, TVD
Standard nonlinear (Newton)and linear (MG) solvers

→ Increases the critical Wi number dramatically !!

Numerical Results (steady problem tests)

- Driven cavity



Velocity profile at the upper wall: $v_{in}\ x^2(1-x)^2$
Dirichtlet Bc‘s everywhere
Stress field: Neuman Bc‘s

$$v_{in} = 16$$

1. 4 to 1 contraction

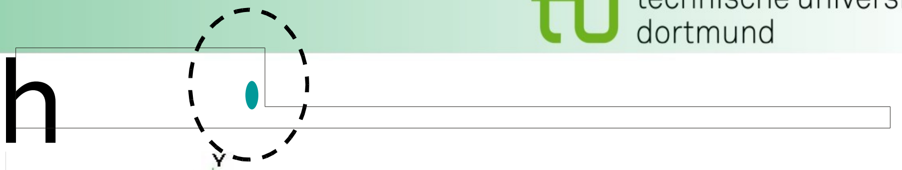
Velocity profile at the inlet: $\frac{3}{128} v_{in}\left(16 - y^2\right)$

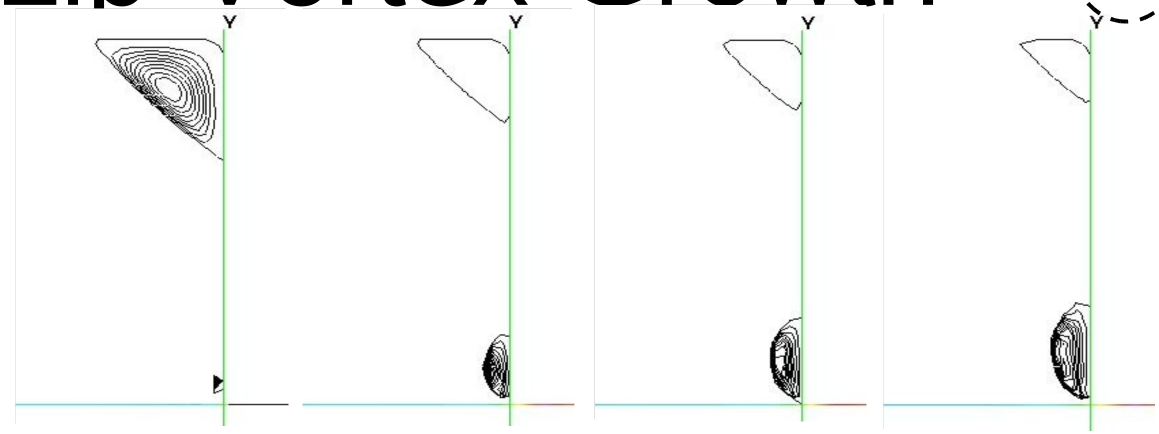
Out flow: Neuman Bc‘s

Stress field: Neuman Bc‘s

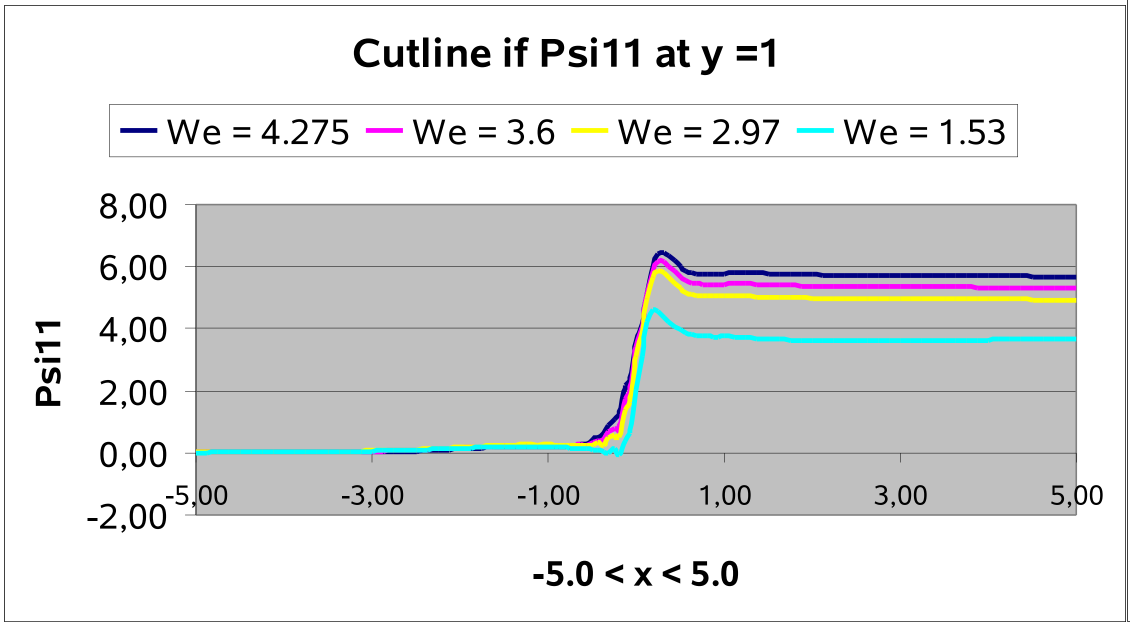
$$v_{in} = 1.0$$

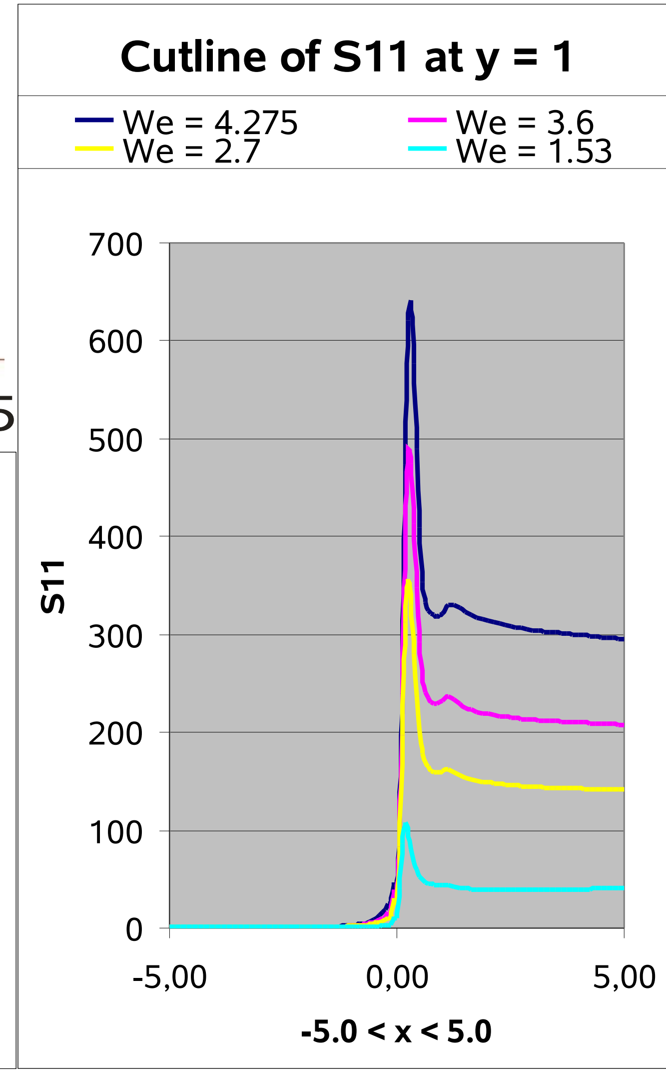
Lip-Vortex Growth





We=1.53 We=2.97 We=3.6 We=4.275





Numerical Results (unsteady problem test

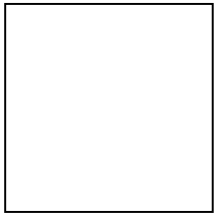
Driven cavity

Velocity profile at the upper wall:

$$v_{in}\, x^2 (1-x)^2$$

$$v_{in} = 8(1+\tanh(8(t-0.5)))$$

For t > 1, v_{in} = 16

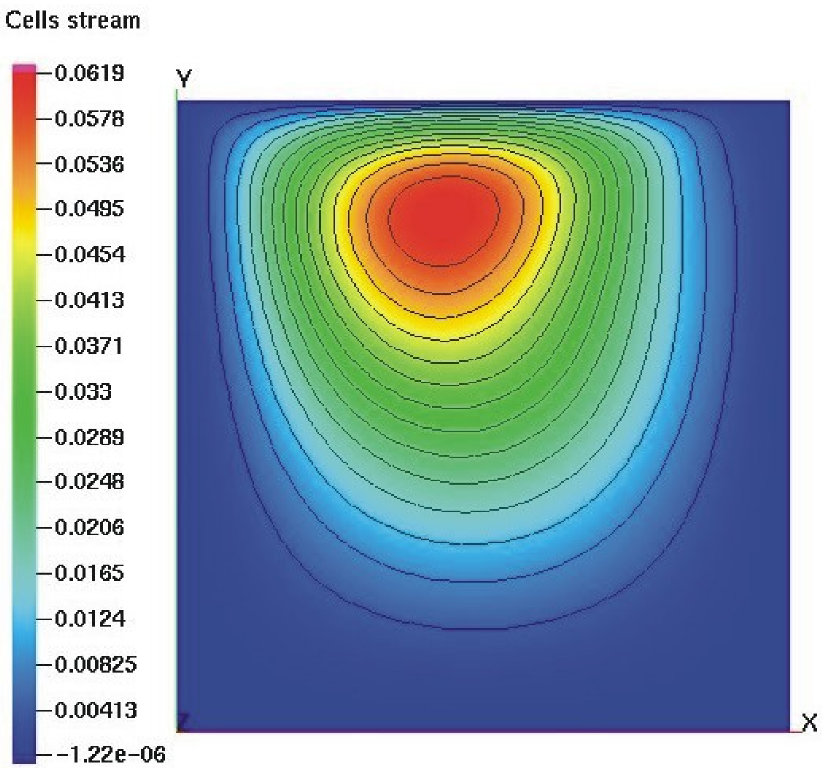


Dirichtlet Bc's everywhere
Stress field: Neuman Bc's

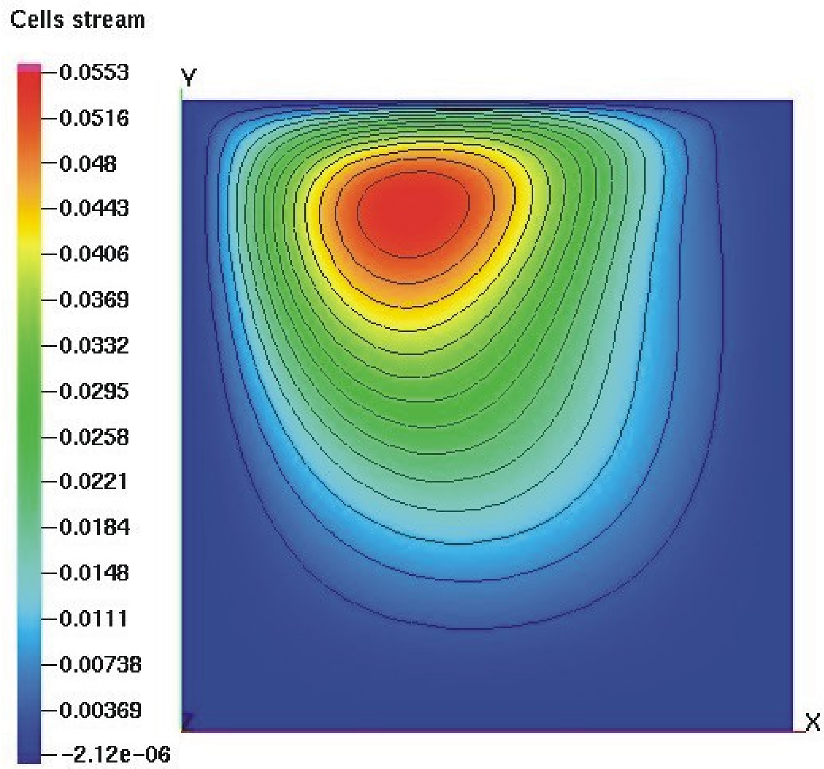
Stream function

t = 8

We = 1



We = 3

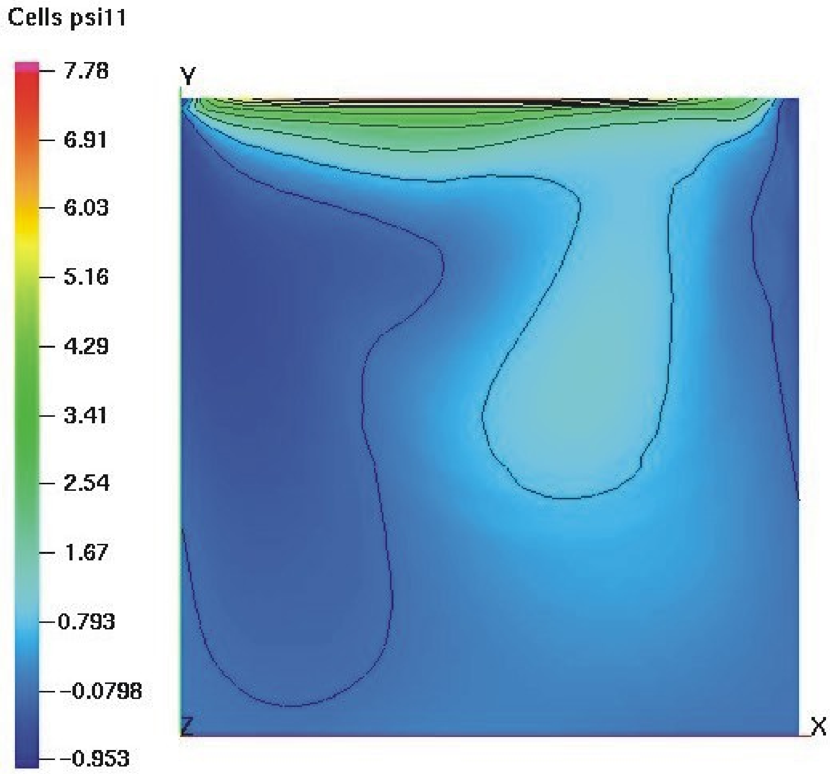


Increasing Wi number shifts the stream to the left

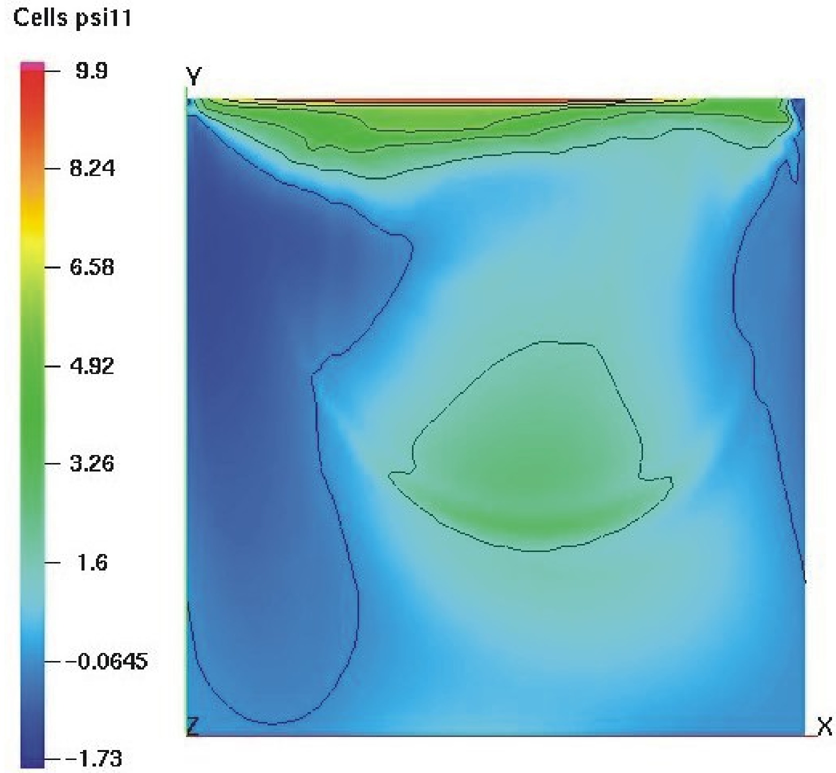
ψ_{11}

t = 8

We = 1



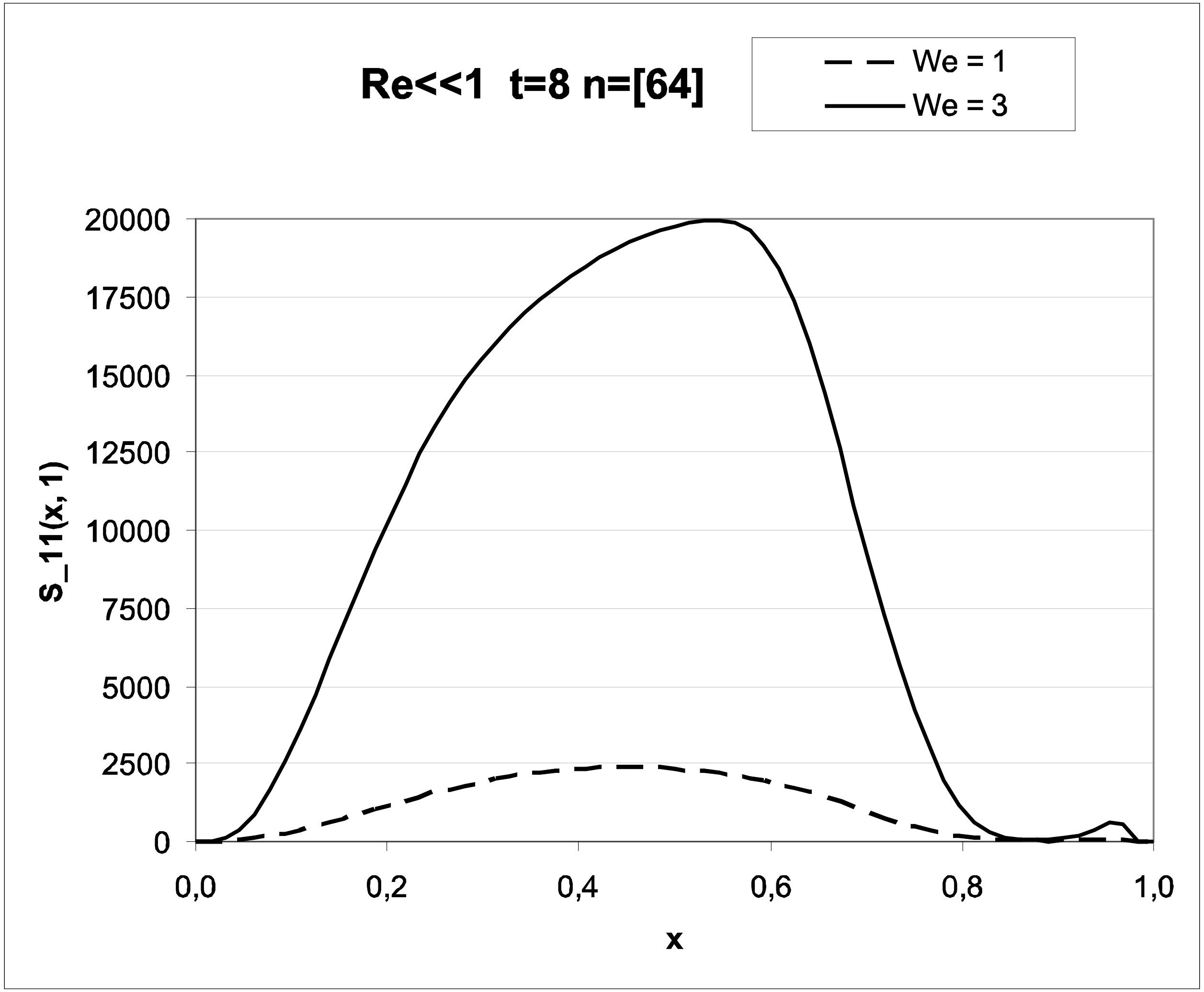
We = 3



Increasing Wi number increases psi by a factor of 1

σ_{11}

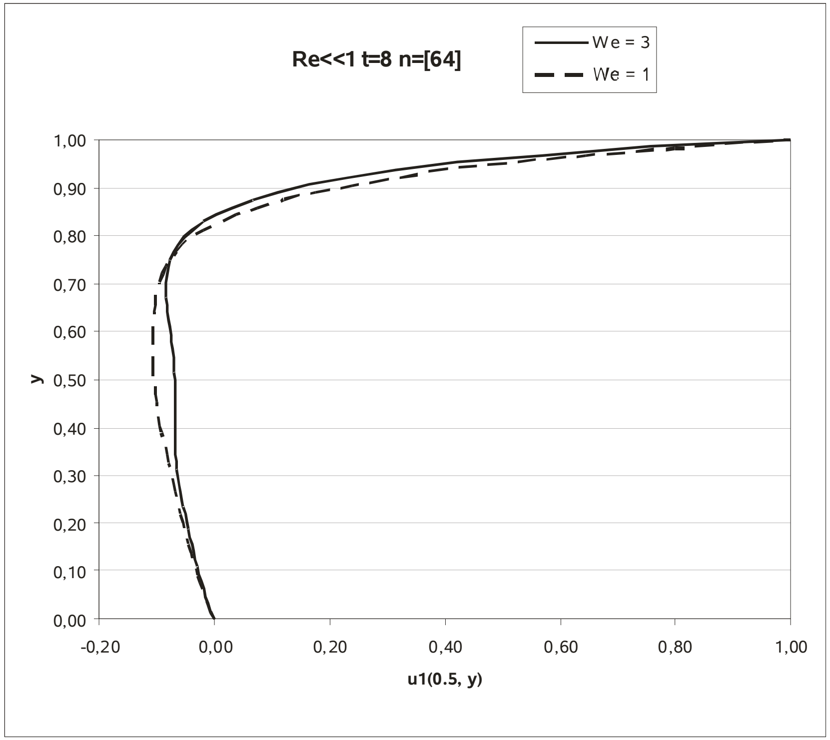
t = 8


Re<<1 t=8 n=[64]
We = 1
We = 3
20000
17500
15000
12500
10000
7500
5000
2500
0
S_11(x, 1)
0,0
0,2
0,4
0,6
0,8
1,0
x

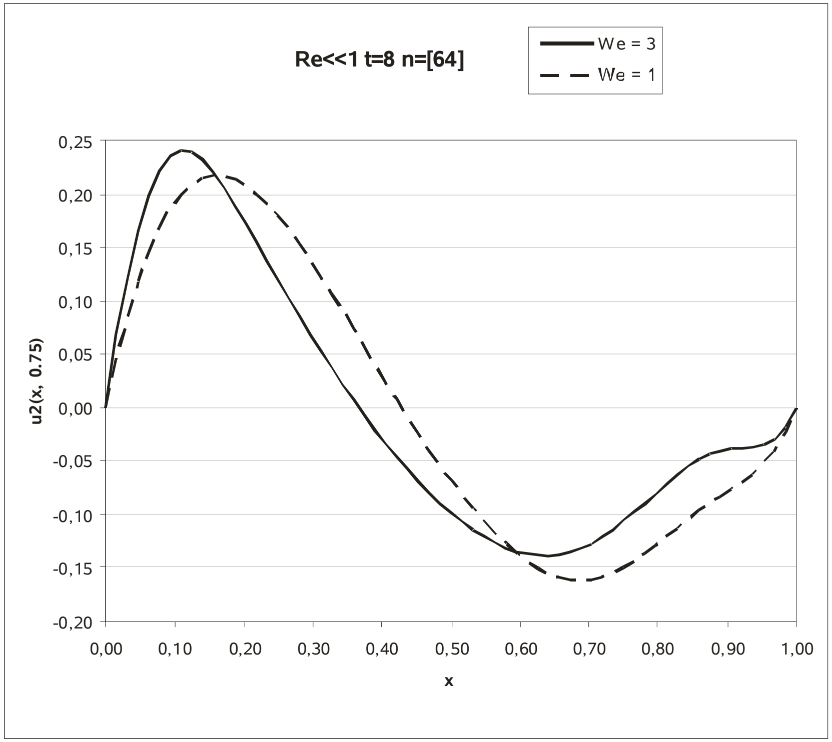
Velocities

t = 8

$V_{x = 0.5}$



$V_{y = 0.75}$



Increasing Wi number does not give much impact to the velocity field

Summary

With LCR, we are now able to simulate much higher Wi numbers

→ **Wi ~ 1.0 for 4 to 1 configuration**

→ **Wi ~ 0.5 for square**

NEW:

→ **Wi >> 4.5 for 4 to 1 contraction (steady state)**

→ **Wi >> 1.5 for square (steady state)**

Additional stabilization will help for high Re + Wi numbers

→ **LCR + Edge Oriented/TVD stabillization**

Application to other viscoelastic flow models