

Berlin, 30.06.2025

## Numerical Mathematics III – Partial Differential Equations

### Exercise Problems 10 (final exercise sheet)

**Attention:** The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

Solving problems 2, 3, and 4 is voluntary. The solutions will not be graded.

1. *Properties of the interpolation operator.* Show that the interpolation operator  $I_{\hat{K}} : C^s(\hat{K}) \rightarrow \hat{P}(\hat{K})$ , which was defined in the lecture, is a linear and continuous operator. The spaces  $C^s(\hat{K})$  and  $P(\hat{K}) \subset C^s(\hat{K})$  are equipped with the norm of  $C^s(\hat{K})$ . In addition, show that the restriction of  $I_{\hat{K}}$  to  $P(\hat{K})$  is the identity operator. **4 points**

2. Let the domain  $\Omega = (0, 1)^2$  be triangulated with the two triangles with the vertices  $(0, 0), (1, 1), (0, 1)$  and  $(0, 0), (1, 0), (1, 1)$ . Compute the error between the function

$$v(x, y) = \sin(2x + y) + 6x^3y^2 \in C^\infty(\Omega)$$

and its interpolant  $Iv$ :

- in the finite element space  $P_0$ , the functional is the value of the function in the barycenter of the triangle, the error should be given in the  $L^2(\Omega)$  norm,
  - in the finite element space  $P_1$ , the functionals are the values of the function in the vertices of the triangles, the error should be given in the  $L^2(\Omega)$  norm and in the  $H^1(\Omega)$  semi norm.
3. *Estimates in the proof of the interpolation estimate of the Clément operator.*
    - i) Show the estimate

$$|p_i(V_i)| \leq ch^{-d/q} \|v\|_{L^q(\omega_i)}.$$

- ii) Prove that one can find a polynomial  $p \in P_1(\omega_K)$  with

$$\|D^j(v - p)\|_{L^q(\omega_K)} \leq Ch^{l-j} \|D^l v\|_{L^q(\omega_K)}, \quad 0 \leq j \leq l \leq 2.$$

4. Consider the function

$$v(x) = \begin{cases} 0.5 & x \in (0, 0.25) \\ 1 & x \in (0.25, 0.5) \\ 0 & x \in (0.5, 0.75) \\ -0.5 & x \in (0.75, 1). \end{cases}$$

Compute the Clément interpolant of this function in the finite element space  $P_1$ , which is defined on the domain  $\Omega = (0, 1)$  and the grid with the nodes  $\{0, 0.25, 0.5, 0.75, 1\}$ .

Hint: the integrals can be computed with Mathematica etc.

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, July 07, 2025, 10:00 a.m.** via the white-board.