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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 07

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

1. *Boundedness of a bilinear form.* Let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ denote the bilinear form

$$a(u,v) = \int_\Omega \nabla u(\boldsymbol{x})^T A(\boldsymbol{x}) \nabla v(\boldsymbol{x}) + c(\boldsymbol{x}) u(\boldsymbol{x}) v(\boldsymbol{x}) \, d\boldsymbol{x},$$

with $a_{ij} \in L^\infty(\Omega)$, $i,j = 1,\ldots,d$, $c \in L^\infty(\Omega)$, $c \geq 0$. Show that this bilinear form is bounded, i.e., there is a constant C such that

$$|a(u,v)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall\, u,v \in H^1(\Omega).$$

2 points

2. *Connection of properties of matrices and bilinear forms.* Let

$$A = (a_{ij}) = a(\phi_j, \phi_i),$$

where $\{\phi_i\}_{i=1}^k$ is the basis of a finite-dimensional space V_k. Show that

i)
$$A = A^T \quad \Longleftrightarrow \quad a(v,w) = a(w,v) \quad \forall\, v,w \in V_k,$$

ii)
$$\underline{v}^T A \underline{v} > 0 \quad \forall\, \underline{v} \in \mathbb{R}^k,\ \underline{v} \neq \underline{0} \quad \Longleftrightarrow \quad a(v,v) > 0 \quad \forall\, v \in V_k,\ v \neq 0.$$

2 points

3. *Stability estimate for the Poisson problem.* Let $\Omega \in \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz boundary and let $f \in L^2(\Omega)$. Consider the Poisson problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

Show the stability estimate

$$\|u\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

2 points

4. *Weak formulation for Robin boundary conditions.* Consider the partial differential equation

$$-\nabla \cdot (A(\boldsymbol{x})\nabla u(\boldsymbol{x})) + c(\boldsymbol{x})u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } \Omega$$

with so-called Robin boundary conditions

$$(A(\boldsymbol{x})\nabla u(\boldsymbol{x})) \cdot \boldsymbol{n}(\boldsymbol{x}) + a(\boldsymbol{x})u(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{on } \partial\Omega.$$

All functions are assumed to be sufficiently regular with $A(\boldsymbol{x}) = A^T(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$. Derive a weak or variational formulation of this problem. **1 point**

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, June 16, 2025, 10:00 a.m.** via the whiteboard.