

Berlin, 26.05.2025

## Numerical Mathematics III – Partial Differential Equations

### Exercise Problems 06

**Attention:** The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

1. *Poincaré–Friedrichs type inequality.* Prove the following inequality of Poincaré–Friedrichs type. Let  $\Omega$  be a bounded domain with Lipschitz boundary and let  $\Omega' \subset \Omega$  with  $\text{meas}_{\mathbb{R}^d}(\Omega') = \int_{\Omega'} d\mathbf{x} > 0$ , then for all  $u \in W^{1,p}(\Omega)$ ,  $p \in [1, \infty)$ , it is

$$\int_{\Omega} |u(\mathbf{x})|^p d\mathbf{x} \leq C \left( \left| \int_{\Omega'} u(\mathbf{x}) d\mathbf{x} \right|^p + \int_{\Omega} \|\nabla u(\mathbf{x})\|_2^p d\mathbf{x} \right),$$

where  $\|\cdot\|_2$  denotes the Euclidean norm of a vector, which is different than the  $l^p$  vector norm for  $p \neq 2$ . **4 points**

2. *Integration by parts.*

- (a) Prove Corollary 3.44: Let the conditions of Theorem 3.42 on the domain  $\Omega$  be satisfied. Consider  $w \in W^{1,p}(\Omega)$  and  $v \in W^{1,q}(\Omega)$  with  $p \in (1, \infty)$  and  $1/p + 1/q = 1$ . Then, it is

$$\int_{\Omega} \partial_i w(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = \int_{\Gamma} w(\mathbf{s}) v(\mathbf{s}) \mathbf{n}_i(\mathbf{s}) d\mathbf{s} - \int_{\Omega} w(\mathbf{x}) \partial_i v(\mathbf{x}) d\mathbf{x}.$$

**2 points**

- (b) Let  $\Omega$  be a bounded Lipschitz domain and let  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ . Prove the interpolation inequality

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq \|u\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)},$$

with

$$\|\nabla u\|_{L^2(\Omega)} = \left( \int_{\Omega} \nabla u \cdot \nabla u d\mathbf{x} \right)^{1/2}.$$

**2 points**

3. *Imbedding of Sobolev spaces with the same order of the derivative  $k$  and different integration powers.* Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain,  $k \geq 0$ , and  $p, q \in [1, \infty]$  with  $q > p$ . Then, it is  $W^{k,q}(\Omega) \subset W^{k,p}(\Omega)$ . **3 points**

4. *Alternative inner product in  $L^2(0, \infty)$ .* Show that

$$a(u, v) = \int_0^\infty e^{-x} u(x) v(x) dx$$

defines a (real) inner product in  $L^2(0, \infty)$ .

Hint: A map  $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  is a (real) inner product if it is bilinear, symmetric, and positive definite, i.e.:

- i)  $a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w)$ ,  $a(u, \alpha v + \beta w) = \alpha a(u, v) + \beta a(u, w)$ ,  $\forall u, v, w \in V, \alpha, \beta \in \mathbb{R}$ ,
- ii)  $a(u, v) = a(v, u) \forall u, v \in V$ ,
- iii)  $a(u, u) \geq 0 \forall u \in V$  and  $a(u, u) = 0 \iff u = 0$ .

**4 points**

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, June 02, 2025, 10:00 a.m.** via the white-board.