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Numerical Mathematics III – Partial Differential Equations

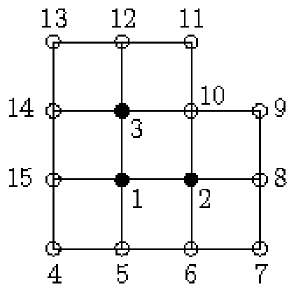
Exercise Problems 03

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

1. *Five point stencil.* Consider the Dirichlet problem for the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

   and the corresponding finite difference discretization with the five point stencil on the following grid:



   Compute the matrices $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 12}$ for the finite difference equation

$$A\underline{u} = \underline{f} + B\underline{g}$$

   with $\underline{u} = (u_1, u_2, u_3)^T$ and $\underline{g} = (u_4, \ldots, u_{15})^T$. **3 points**

2. *Comparison lemma.* Prove the comparison lemma (Corollary 2.24). **3 points**

3. Let the assumptions of the discrete maximum principle for the boundary value problem be satisfied. Prove the following stability estimate: For the solution of the problem

$$\begin{aligned} L_h u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \omega_h, \\ u(\mathbf{x}) &= 0, \quad \mathbf{x} \in \gamma_h, \end{aligned}$$

   with $d(\mathbf{x}) > 0$ for all $\mathbf{x} \in \omega_h$, it is

$$\|u\|_{l^\infty(\omega_h \cup \gamma_h)} \le \left\|D^{-1} f\right\|_{l^\infty(\omega_h)}$$

   with $D = \operatorname{diag}(d(\mathbf{x}))$ for $\mathbf{x} \in \omega_h$. The notations and the properties of the individual terms of the finite difference operator are the same as in the lecture, Section 2.3. **3 points**

4. *An eigenvalue problem connected to the five point stencil.* Show that the vector $v_k = (v_{k,0}, \cdots, v_{k,n})$ with

$$v_{k,0} = v_{k,n} = 0, \quad v_{k,i} = \sqrt{2}\, \sin\left(\pi\, k\, x_i\right),$$

solves the eigenvalue problem

$$v_{k,i-1} + \left(\lambda_k\, h^2 - 2\right) v_{k,i} + v_{k,i+1} = 0$$

with

$$\lambda_k = \frac{2}{h^2}\left(1 - \cos\left(\pi\, k\, h\right)\right) = \frac{4}{h^2} \sin^2\left(\frac{\pi\, k\, h}{2}\right).$$

2 points

5. Do not forget the programming problem from exercise sheet 02.

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, May 12, 2025, 10:00 a.m.** via the whiteboard.