Department for Mathematics and Computer Science Freie Universität Berlin Prof. Dr. V. John, john@wias-berlin.de André-Alexander Zepernick, a.zepernick@fu-berlin.de

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Numerical Mathematics III – Partial Differential Equations Exercise Problems 02

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

- 1. Basic properties of finite difference approximations. Solve the following problems.
 - i) Show that

$$v_{\bar{x},i} = \frac{1}{2} \left(v_{x,i} + v_{\bar{x},i} \right), \quad v_{\bar{x}x,i} = \left(v_{\bar{x},i} \right)_{x,i}.$$

1 point

ii) Consider a function v(x) at x_i and show the following consistency estimates

$$v_{\bar{x},i} = v'(x_i) + \mathcal{O}(h^2), \quad v_{\bar{x}x,i} = v''(x_i) + \mathcal{O}(h^2).$$

1 point

iii) Compute the order of consistency of the following finite difference approximation

$$u''(x) \sim \frac{1}{12h^2} \Big(-u(x+2h) + 16u(x+h) - 30u(x) + 16u(x-h) - u(x-2h) \Big).$$

2 points

4 points

2. Finite difference approximation of the second order derivative for non-constant diffusion. Consider the differential operator $Lu = \frac{\partial}{\partial x} \left(k\left(x\right) \frac{\partial u}{\partial x} \right)$ and its finite difference approximation

$$(L_h u_h)_i = \frac{1}{h} \left(a_{i+1} \frac{u_{i+1} - u_i}{h} - a_i \frac{u_i - u_{i-1}}{h} \right).$$

Show that $a_i = \frac{k_i + k_{i-1}}{2}$ and $a_i = k \left(x_i - \frac{h}{2} \right)$ satisfy the conditions for second order consistency

$$\frac{a_{i+1} - a_i}{h} = k'(x_i) + \mathcal{O}(h^2), \qquad \frac{a_{i+1} + a_i}{2} = k(x_i) + \mathcal{O}(h^2),$$

which were derived in the lecture.

Finite difference approximation of the second order derivative at a non-equidistant grid. Consider the interval [x − h_x⁻, x + h_x⁺] with h_x⁻, h_x⁺ > 0, h_x⁻ ≠ h_x⁺.

i) Assume $u \in C^3([x - h_x^-, x + h_x^+])$. Show the following consistency estimate

$$\left| u''(x) - \frac{2}{h_x^+ + h_x^-} \left(\frac{u(x+h_x^+) - u(x)}{h_x^+} - \frac{u(x) - u(x-h_x^-)}{h_x^-} \right) \right|$$

$$\leq C \left(h_x^+ + h_x^- \right).$$

3 points

ii) Prove that there is no other approximation which satisfies

$$|u''(x) - (\alpha u (x - h_x^{-}) + \beta u (x) + \gamma u (x + h_x^{+}))| \le C (h_x^{+} + h_x^{-}),$$

where the weights depend only on the mesh width, i.e., $\alpha = \alpha (h_x^-, h_x^+)$, $\beta = \beta (h_x^-, h_x^+)$, $\gamma = \gamma (h_x^-, h_x^+) \in \mathbb{R}$, and the constant *C* does not explicitly depend on $u(\mathbf{x})$, $u'(\mathbf{x})$, and $u''(\mathbf{x})$. **2** points

4. This problem has to be solved until May 12, 2025!

Code for five point stencil. Write a code for the numerical solution of

$$-\Delta u = f \quad \text{in } \Omega = (0,1)^2$$
$$u = q \quad \text{on } \partial \Omega.$$

The right-hand side and the boundary conditions should be chosen such that

$$u(x,y) = x^4 y^5 - 17\sin(xy)$$

is the solution of the boundary value problem.

Use the five point stencil for the discretizing the partial differential equation. The mesh widths should be chosen to be

$$h_x = h_y = h = 2^{-n}$$
 $n = 2, 3, 4, \dots, 8.$

Order the nodes lexicographically and store the matrix in sparse format. 6 points



Compute the following errors

 $||u - u_h||_{l^{\infty}(\omega_h)}, \qquad ||u - u_h||_{l^2(\omega_h)},$

and the orders of convergence, based on the errors on the two finest meshes, where the second norm is the standard Euclidean vector norm. 2 points

The exercise problems should be solved in groups of four students. The solutions have to be submitted until Monday, May 05, 2025, 10:00 a.m. via the white-board.