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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 01

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

1. *Basic properties of the nabla operator.* The following operators are defined for a scalar function u a and a vector-valued function $\mathbf{v} = (v_1, v_2, v_3)^T$ with the help of the nabla operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)^T = (\partial_{x_1}, \cdots, \partial_{x_d}) :$$

- $\mathrm{grad} u = \nabla u = (\partial_x u, \partial_y u, \partial_z u)^T$,
- $\mathrm{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3$,
- $\mathrm{rot} \mathbf{v} = \nabla \times \mathbf{v} = \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix}$.

Assuming that all considered functions are sufficiently smooth (sufficiently often differentiable), show the following identities:

i) $\nabla \cdot \nabla u = \Delta u = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u$,

ii) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$,

iii) $\nabla \times (\nabla u) = 0$,

iv) $\nabla \times (u\mathbf{v}) = u\,(\nabla \times \mathbf{v}) - (\mathbf{v} \times \nabla u)$,

v) $\nabla \cdot (u\mathbf{v}) = \nabla u \cdot \mathbf{v} + u\, \nabla \cdot \mathbf{v}$.

4 points

2. *Analytic solution of a one-dimensional heat equation.* Show that

$$u\,(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0\,(z)\, \mathrm{e}^{\frac{-(x-z)^2}{4t}}\, dz$$

is a solution of the one-dimensional heat equation

$$\begin{aligned} \partial_t u - \partial_{xx} u &= 0, & x \in \mathbb{R}, t > 0, \\ u\,(0, x) &= u_0\,(x)\,, & x \in \mathbb{R}. \end{aligned}$$

It shall be assumed that $u_0(x)$ is sufficiently smooth.
<u>Hint.</u> To check the initial condition, assume that $u_0(x)$ can be expanded in a uniformly convergent Fourier series

$$u_0(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos\,(n\omega x) + \beta_n \sin\,(n\omega x)\right),$$

with $\omega \in \mathbb{R}$, and use the following identities

$$\begin{aligned} \int_{-\infty}^{\infty} \sin(nz)\, \mathrm{e}^{\frac{-(x-z)^2}{4t}}\; dz &= \sqrt{4\pi t}\mathrm{e}^{-n^2 t} \sin(nx)\,, \\ \int_{-\infty}^{\infty} \cos(nz)\, \mathrm{e}^{\frac{-(x-z)^2}{4t}}\; dz &= \sqrt{4\pi t}\mathrm{e}^{-n^2 t} \cos(nx)\,, \quad n \in \mathbb{R}. \end{aligned}$$

4 points

3. *Classification of second order partial differential equations.* Classify the following partial differential equations

$$\begin{aligned} &\text{i)} & \partial_{xx}u + 2\partial_{xy}u + 2\partial_{yy}u + 4\partial_{yz}u + 5\partial_{zz}u + \partial_x u + \partial_y u &= 0, \quad (3d)\,, \\ &\text{ii)} & e^z \partial_{xy}u - \partial_{xx}u - \log\left(x^2 + y^2 + z^2\right) &= 0, \quad (3d)\,, \\ &\text{iii)} & \partial_{xx}u + 4\partial_{xy}u + 3\partial_{yy}u + 3\partial_x u - \partial_y u + 2u &= 0, \quad (2d)\,, \\ &\text{iv)} & a\partial_{xx}u + 2a\partial_{xy}u + a\partial_{yy}u + b\partial_x u + c\partial_y u + u &= 0, \quad (2d)\,, \end{aligned}$$

with $a \neq 0$ in iv) and $2d$ – in two dimensions, $3d$ – in three dimensions.

4 points

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, April 28, 2025, 10:00 a.m.** via the whiteboard.