Chapter 2
Finite Difference Methods for Elliptic Equations

Remark 2.1. Model problem. The model problem in this chapter is the Poisson equation with Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$. This chapter follows in wide parts Samarskij (1984). □

2.1 Basics on Finite Differences

Remark 2.2. Grid. This section considers the one-dimensional situation. Consider the interval $[0,1]$ that is decomposed by an equidistant grid

$$\begin{aligned} x_i &= ih, \quad i = 0, \ldots, n, \quad h = 1/n, \text{ – nodes}, \\ \omega_h &= \{x_i \ : \ i = 0, \ldots, n\} \text{ – grid}. \end{aligned}$$

□

Definition 2.3. Grid function. A vector $\underline{u}_h = (u_0, \ldots, u_n)^T \in \mathbb{R}^{n+1}$ that assigns every grid point a function value is called grid function. □

Definition 2.4. Finite differences. Let $v(x)$ be a sufficiently smooth function and denote by $v_i = v(x_i)$, where x_i are the nodes of the grid. The following quotients are called

$$\begin{aligned} v_{x,i} &= \frac{v_{i+1} - v_i}{h} \text{ – forward difference}, \\ v_{\overline{x},i} &= \frac{v_i - v_{i-1}}{h} \text{ – backward difference}, \\ v_{\mathring{x},i} &= \frac{v_{i+1} - v_{i-1}}{2h} \text{ – central difference}, \end{aligned}$$

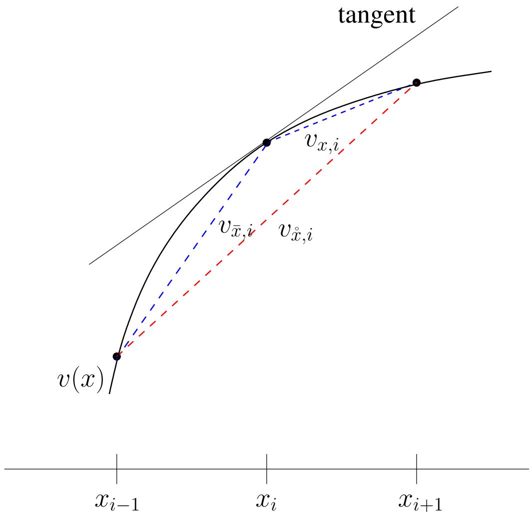

Fig. 2.1 Illustration of the finite differences.

$$v_{\overline{x}x,i} = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} \text{ – second order difference,}$$

see Figure 2.1. □

Remark 2.5. Some properties of the finite differences. It is (*exercise*)

$$v_{\mathring{x},i} = \frac{1}{2}(v_{x,i} + v_{\overline{x},i}), \quad v_{\overline{x}x,i} = (v_{\overline{x},i})_{x,i}.$$

Using the Taylor series expansion for $v(x)$ at the node x_i, one gets (*exercise*)

$$\begin{aligned}
v_{x,i} &= v'(x_i) + \frac{1}{2}hv''(x_i) + \mathcal{O}\left(h^2\right), \\
v_{\overline{x},i} &= v'(x_i) - \frac{1}{2}hv''(x_i) + \mathcal{O}\left(h^2\right), \\
v_{\mathring{x},i} &= v'(x_i) + \mathcal{O}\left(h^2\right), \\
v_{\overline{x}x,i} &= v''(x_i) + \mathcal{O}\left(h^2\right).
\end{aligned}$$

□

Definition 2.6. Consistent difference operator. Let L be a differential operator. The difference operator $L_h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is called consistent with L of order k if

$$\max_{0 \le i \le n} |(Lu)(x_i) - (L_h u_h)_i| = \|Lu - L_h u_h\|_{\infty,\omega_h} = \mathcal{O}\left(h^k\right)$$

for all sufficiently smooth functions $u(x)$. □

Example 2.7. Consistency orders. The order of consistency measures the quality of approximation of L by L_h.

The difference operators $v_{x,i}, v_{\overline{x},i}, v_{\mathring{x},i}$ are consistent to $L = \frac{d}{dx}$ with order $1, 1$, and 2, respectively. The operator $v_{\overline{x}x,i}$ is consistent of second order to $L = \frac{d^2}{dx^2}$, see Remark 2.5. □

Example 2.8. Approximation of a more complicated differential operator by difference operators. Consider the differential operator

$$Lu = \frac{d}{dx}\left(k(x)\frac{du}{dx}\right),$$

where $k(x)$ is assumed to be continuously differentiable. Define the difference operator L_h as follows

$$\begin{aligned}(L_h u_h)_i = (a u_{\overline{x},i})_{x,i} &= \frac{1}{h}\Big(a(x_{i+1})u_{\overline{x},i}(x_{i+1}) - a(x_i)u_{\overline{x},i}(x_i)\Big) \\ &= \frac{1}{h}\left(a_{i+1}\frac{u_{i+1}-u_i}{h} - a_i\frac{u_i - u_{i-1}}{h}\right), \end{aligned} \tag{2.2}$$

where a is a grid function that has to be determined appropriately. One gets with the product rule

$$(Lu)_i = k'(x_i)(u')_i + k(x_i)(u'')_i$$

and with a Taylor series expansion for u_{i-1}, u_{i+1}, which is inserted in (2.2),

$$(L_h u_h)_i = \frac{a_{i+1}-a_i}{h}(u')_i + \frac{a_{i+1}+a_i}{2}(u'')_i + \frac{h(a_{i+1}-a_i)}{6}(u''')_i + \mathcal{O}\left(h^2\right).$$

Thus, the difference of the differential operator and the difference operator is

$$\begin{aligned}(Lu)_i - (L_h u_h)_i = &\left(k'(x_i) - \frac{a_{i+1}-a_i}{h}\right)(u')_i + \left(k(x_i) - \frac{a_{i+1}+a_i}{2}\right)(u'')_i \\ &- \frac{h(a_{i+1}-a_i)}{6}(u''')_i + \mathcal{O}\left(h^2\right). \end{aligned} \tag{2.3}$$

In order to define L_h so that it is consistent of second order to L, one has to satisfy the following two conditions

$$\frac{a_{i+1}-a_i}{h} = k'(x_i) + \mathcal{O}\left(h^2\right), \quad \frac{a_{i+1}+a_i}{2} = k(x_i) + \mathcal{O}\left(h^2\right).$$

From the first requirement, it follows that $a_{i+1} - a_i = \mathcal{O}(h)$. Hence, the third term in the consistency error equation (2.3) is of order $\mathcal{O}\left(h^2\right)$. Possible choices for the grid function are (*exercise*)

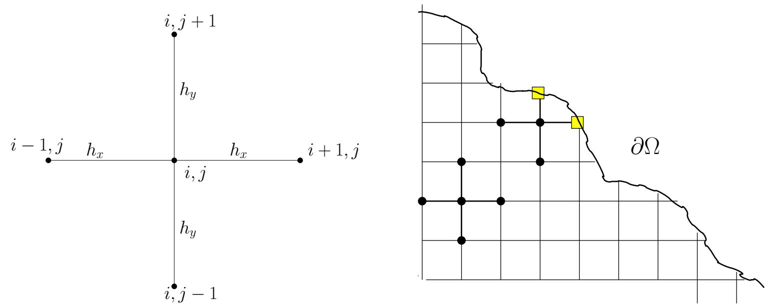

Fig. 2.2 Five point stencils.

$$a_i = \frac{k_i + k_{i-1}}{2}, \quad a_i = k\left(x_i - \frac{h}{2}\right), \quad a_i = (k_i k_{i-1})^{1/2}.$$

Note that the 'natural' choice, $a_i = k_i$, leads only to first order consistency. (*exercise*) □

2.2 Finite Difference Approximation of the Laplacian in Two Dimensions

Remark 2.9. The five point stencil. The Laplacian in two dimensions is defined by

$$\Delta u(\boldsymbol{x}) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \partial_{xx} u + \partial_{yy} u = u_{xx} + u_{yy}, \quad \boldsymbol{x} = (x, y).$$

The simplest approximation uses for both second order derivatives the second order differences. One obtains the so-called five point stencil and the approximation

$$\begin{aligned}(\Delta u)_{ij} \approx (\Lambda u)_{ij} &= u_{\overline{x}x,i} + u_{\overline{y}y,j} \\ &= \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h_y^2},\end{aligned} \tag{2.4}$$

see Figure 2.2. From the consistency order of the second order difference, it follows immediately that Λu approximates the Laplacian of order $\mathcal{O}\left(h_x^2 + h_y^2\right)$. □

Remark 2.10. The five point stencil on curvilinear boundaries. There is a difficulty if the five point stencil is used in domains with curvilinear boundaries. The approximation of the second derivative requires three function values in each coordinate direction

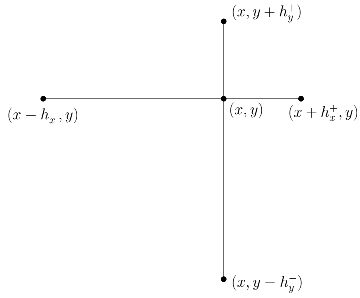

Fig. 2.3 Sketch to Remark 2.10.

$$\begin{aligned}&(x - h_x^-, y), (x, y), (x + h_x^+, y),\\&(x, y - h_y^-), (x, y), (x, y + h_y^+),\end{aligned}$$

see Figure 2.3. A guideline of defining the approximation is that the five point stencil is recovered in the case $h_x^- = h_x^+$ and $h_y^- = h_y^+$. Consider just the x-direction. A possible approximation is

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{1}{\overline{h}_x}\left(\frac{u(x + h_x^+, y) - u(x, y)}{h_x^+} - \frac{u(x, y) - u(x - h_x^-, y)}{h_x^-}\right) \tag{2.5}$$

with $\overline{h}_x = (h_x^+ + h_x^-)/2$. Using a Taylor series expansion, one finds that the error of this approximation is

$$\begin{aligned}&\frac{\partial^2 u}{\partial x^2} - \frac{1}{\overline{h}_x}\left(\frac{u(x + h_x^+, y) - u(x, y)}{h_x^+} - \frac{u(x, y) - u(x - h_x^-, y)}{h_x^-}\right)\\&\qquad = -\frac{1}{3}(h_x^+ - h_x^-)\frac{\partial^3 u}{\partial x^3} + \mathcal{O}\left(\overline{h}_x^2\right).\end{aligned}$$

For $h_x^+ \neq h_x^-$, this approximation is of first order.

A different way consists in using

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{1}{\tilde{h}_x}\left(\frac{u(x + h_x^+, y) - u(x, y)}{h_x^+} - \frac{u(x, y) - u(x - h_x^-, y)}{h_x^-}\right)$$

with $\tilde{h}_x = \max\{h_x^+, h_x^-\}$. However, this approximation possesses only the order zero, i.e., there is actually no approximation.

Altogether, there is a loss of order of consistency at curvilinear boundaries. □

Example 2.11. The Dirichlet problem. Consider the Poisson equation that is equipped with Dirichlet boundary conditions (2.1). First, $\mathbb{R}^2$ is decomposed by a grid with rectangular mesh cells $x_i = ih_x, y_j = jh_y$, $h_x, h_y > 0$, $i, j \in \mathbb{Z}$. Denote by

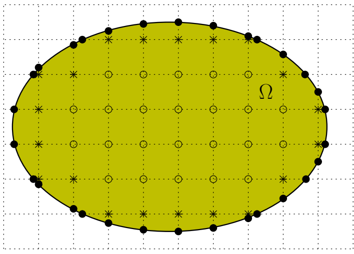

Fig. 2.4 Different types of nodes in the grid.

$w_h^\circ = \{\circ\}$	inner nodes, five point stencil does not contain any boundary node,
$w_h^* = \{*\}$	inner nodes that are close to the boundary, five point stencil contains boundary nodes,
$\gamma_h = \{\bullet\}$	boundary nodes,
$\omega_h = w_h^\circ \cup w_h^*$	inner nodes,
$\omega_h \cup \gamma_h$	grid,

see Figure 2.4.

The finite difference approximation of problem (2.1) that will be studied in the following consists in finding a mesh function $u(\boldsymbol{x})$ such that

$$\begin{aligned} -\Lambda u(\boldsymbol{x}) &= \phi(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in w_h^\circ, \\ -\Lambda^* u(\boldsymbol{x}) &= \phi(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in w_h^*, \\ u(\boldsymbol{x}) &= g(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \gamma_h, \end{aligned} \tag{2.6}$$

where $\phi(\boldsymbol{x})$ is a grid function that approximates $f(\boldsymbol{x})$ and Λ^* is an approximation of the Laplacian for nodes that are close to the boundary, e.g., defined by (2.5). The discrete problem is a large sparse linear system of equations. The most important questions are:

- Which properties possesses the solution of (2.6)?
- Converges the solution of (2.6) to the solution of the Poisson problem and if yes, with which order in the norm $\|\cdot\|_{\infty,\omega_h}$?

□

2.3 The Discrete Maximum Principle for a Finite Difference Approximation

Remark 2.12. Contents of this section. Solutions of the Laplace equation, i.e., of (2.1) with $f(\boldsymbol{x}) = 0$, fulfill so-called maximum principles. This section shows that the finite difference approximation of this operator, where the five point stencil of the Laplacian is a special case, satisfies a discrete analog of one of the maximum principles, under an assumption on the grid. The analysis proceeds along the classical lines, see Samarskij (1984) or (Samarskii, 2001, Chapter 4) □

Theorem 2.13. Maximum principles for harmonic functions. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 1$, be a bounded domain and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω, i.e., $u(\boldsymbol{x})$ solves the Laplace equation $-\Delta u = 0$ in Ω.*

- *Weak maximum principle. It holds*

$$\max_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x}).$$

  That means, $u(\boldsymbol{x})$ takes its maximal value at the boundary.
- *Strong maximum principle. If Ω is connected and if the maximum is taken in Ω (note that Ω is open), i.e., $u(\boldsymbol{x}_0) = \max_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x})$ for a point $\boldsymbol{x}_0 \in \Omega$, then $u(\boldsymbol{x})$ is constant*

$$u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x}) = u(\boldsymbol{x}_0) \quad \forall\ \boldsymbol{x} \in \overline{\Omega}.$$

Proof. See the literature, e.g., (Evans, 2010, p. 27, Theorem 4) or the course on the theory of partial differential equations. ■

Remark 2.14. Interpretation of the maximum principle.

- The Laplace equation models the temperature distribution of a heated body without heat sources in Ω. Then, the weak maximum principle just states that the temperature in the interior of the body cannot be higher than the highest temperature at the boundary.
- There are maximum principles also for more complicated operators than the Laplacian, e.g., see Evans (2010).
- Since the solution of boundary value problems with partial differential equations will be only approximated by a discretization like a finite difference method, one has to expect that basic physical properties are satisfied by the numerical solution also only approximately. However, in applications, it is often very important that such properties are satisfied exactly.

□

Remark 2.15. The difference equation. In this section, a difference equation of the form

$$a(\boldsymbol{x})u(\boldsymbol{x}) = \sum_{\boldsymbol{y}\in S(\boldsymbol{x})} b(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y}) + F(\boldsymbol{x}),\ \boldsymbol{x} \in \omega_h \cup \gamma_h, \qquad (2.7)$$

will be considered. In (2.7), for each node $\boldsymbol{x}$, the set $S(\boldsymbol{x})$ is the set of all nodes on which the sum has to be performed, but $\boldsymbol{x} \notin S(\boldsymbol{x})$. That means, $a(\boldsymbol{x})$ describes the contribution of the finite difference scheme of a node $\boldsymbol{x}$ to itself and $b(\boldsymbol{x},\boldsymbol{y})$ describes the contributions from the neighbors. The algebraic formulation of (2.7) is a linear system of equations. Then, the diagonal entries

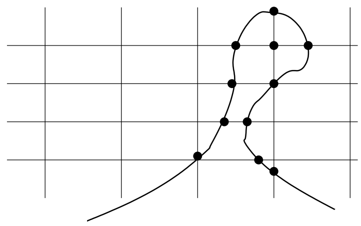

Fig. 2.5 Grid that is not allowed in Section 2.3.

are determined by $a(\boldsymbol{x})$ and the off-diagonal entries by $-b(\boldsymbol{x}, \boldsymbol{y})$, where the minus sign occurs because the term with $b(\boldsymbol{x}, \boldsymbol{y})$ is on the right-hand side of (2.7).

It will be assumed that the grid ω_h of inner nodes is connected, i.e., for all $\boldsymbol{x}_a, \boldsymbol{x}_e \in \omega_h$ exist $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \omega_h$ with $\boldsymbol{x}_1 \in S(\boldsymbol{x}_a), \boldsymbol{x}_2 \in S(\boldsymbol{x}_1), \ldots, \boldsymbol{x}_e \in S(\boldsymbol{x}_m)$. For instance, the situation depicted in Figure 2.5 is not allowed. The algebraic interpretation of this assumption, together with (2.8) below, is that the restriction of the system matrix to the inner nodes is an irreducible matrix.

It will be assumed that the coefficients $a(\boldsymbol{x})$ and $b(\boldsymbol{x}, \boldsymbol{y})$ satisfy the following conditions:

$$\begin{aligned} &a(\boldsymbol{x}) > 0,\ b(\boldsymbol{x}, \boldsymbol{y}) > 0,\ \forall\ \boldsymbol{x} \in \omega_h, \forall\ \boldsymbol{y} \in S(\boldsymbol{x}), \\ &a(\boldsymbol{x}) = 1,\ b(\boldsymbol{x}, \boldsymbol{y}) = 0\ \forall\ \boldsymbol{x} \in \gamma_h \text{ (Dirichlet boundary condition).} \end{aligned} \tag{2.8}$$

The values of the Dirichlet boundary condition are incorporated in (2.7) in the function $F(\boldsymbol{x})$. Thus, the linear system of equations will have the form

$$\begin{pmatrix} A_1 & A_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{u_g} \end{pmatrix} = \begin{pmatrix} \underline{\phi} \\ \underline{g} \end{pmatrix}, \tag{2.9}$$

where I is the identity matrix, $\underline{u}$ is the vector that corresponds to the inner nodes, $\underline{u_g}$ the vector for the boundary nodes, $\underline{\phi}$ the vector for the right-hand side in the inner nodes, and $\underline{g}$ the vector from the given boundary conditions. The matrix block A_1 contains the connections among the inner nodes and the block A_2 the connections of the inner nodes close to the boundary to the boundary nodes. □

Example 2.16. Five point stencil for approximating the Laplacian. Inserting the approximation of the Laplacian with the five point stencil (2.4) for $\boldsymbol{x} = (x, y) \in \omega_h^\circ$ in scheme (2.7) gives

$$\frac{2(h_x^2 + h_y^2)}{h_x^2 h_y^2} u(x, y) = \left[\frac{1}{h_x^2} u(x + h_x, y) + \frac{1}{h_x^2} u(x - h_x, y) \right.$$

$$+\frac{1}{h_y^2}u(x,y+h_y)+\frac{1}{h_y^2}u(x,y-h_y)\Bigg]+\phi(x,y).$$

It follows that

$$\begin{aligned}
a(\boldsymbol{x}) &= \frac{2(h_x^2+h_y^2)}{h_x^2h_y^2} > 0,\\
b(\boldsymbol{x},\boldsymbol{y}) &\in \{h_x^{-2},h_y^{-2}\} > 0,\\
S(\boldsymbol{x}) &= \{(x-h_x,y),(x+h_x,y),(x,y-h_y),(x,y+h_y)\}.
\end{aligned}$$

For inner nodes that are close to the boundary, only the one-dimensional case (2.5) will be considered for simplicity. Let $x+h_x^+\in\gamma_h$, then it follows by inserting (2.5) in (2.7)

$$\frac{1}{\overline{h}_x}\left(\frac{1}{h_x^+}+\frac{1}{h_x^-}\right)u(x,y)=\frac{u(x-h_x^-,y)}{\overline{h}_xh_x^-}+\underbrace{\frac{u(x+h_x^+,y)}{\overline{h}_xh_x^+}}_{\text{on }\gamma_h\to A_2}+\phi(x), \qquad (2.10)$$

such that

$$\begin{aligned}
a(x) &= \frac{1}{\overline{h}_x}\left(\frac{1}{h_x^+}+\frac{1}{h_x^-}\right) > 0,\\
b(x,y) &\in \left\{\frac{1}{\overline{h}_xh_x^-},\frac{1}{\overline{h}_xh_x^+}\right\} > 0,\\
S(x) &= \{(x-h_x^-,y),(x+h_x^+,y)\}.
\end{aligned}$$

Hence, the assumptions (2.8) on the coefficients are satisfied. □

Remark 2.17. Reformulation of the difference scheme. Scheme (2.7) can be reformulated in the form

$$d(\boldsymbol{x})u(\boldsymbol{x})=\sum_{\boldsymbol{y}\in S(\boldsymbol{x})}b(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{y})-u(\boldsymbol{x})\big)+F(\boldsymbol{x}) \qquad (2.11)$$

with $d(\boldsymbol{x})=a(\boldsymbol{x})-\sum_{\boldsymbol{y}\in S(\boldsymbol{x})}b(\boldsymbol{x},\boldsymbol{y})$. Algebraically, $d(\boldsymbol{x})$ is the sum of the matrix entries of the row that corresponds to the node $\boldsymbol{x}$. □

Example 2.18. Five point stencil for approximating the Laplacian. Using the five point stencil for approximating the Laplacian, form (2.11) of the scheme is obtained with

$$d(\boldsymbol{x})=\frac{2(h_x^2+h_y^2)}{h_x^2h_y^2}-\frac{2}{h_x^2}-\frac{2}{h_y^2}=0 \qquad (2.12)$$

for $\boldsymbol{x}\in\omega_h^\circ$. Thus, the corresponding row sums of the matrix are zero.

For nodes close to the boundary $\boldsymbol{x}\in\omega_h^*$, again only the one-dimensional situation as in Example 2.16 is considered. One obtains

$$d(x) = \frac{1}{\overline{h}_x}\left(\frac{1}{h_x^+} + \frac{1}{h_x^-}\right) - \frac{1}{\overline{h}_x h_x^-} - \frac{1}{\overline{h}_x h_x^+} = 0,$$

i.e., also for such nodes, the corresponding row sum vanishes.

The coefficients $a(\boldsymbol{x})$ and $b(\boldsymbol{x},\boldsymbol{y})$ are the weights of the finite difference stencil for approximating the Laplacian. A minimal condition for consistency is that this approximation vanishes for constant functions since the derivatives of constant functions vanish. The algebraic formulation of this consistency condition is just that all row sums vanish, since a constant function is represented by a constant vector. If the row sums vanish, then the multiplication of the matrix with a constant vector gives the zero vector. □

Lemma 2.19. Discrete maximum principle (DMP) for inner nodes. *Let $u(\boldsymbol{x}) \neq const$ on ω_h and $d(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in \omega_h$. Then, it follows from*

$$L_h u(\boldsymbol{x}) := d(\boldsymbol{x})u(\boldsymbol{x}) - \sum_{\boldsymbol{y}\in S(\boldsymbol{x})} b(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{y}) - u(\boldsymbol{x})\big) \le 0 \tag{2.13}$$

(or $L_h u(\boldsymbol{x}) \ge 0$, respectively) on ω_h that $u(\boldsymbol{x})$ does not possess a positive maximum (or negative minimum, respectively) on ω_h.

Proof. The proof is performed by contradiction. Let $L_h u(\boldsymbol{x}) \le 0$ for all $\boldsymbol{x} \in \omega_h$ and assume that $u(\boldsymbol{x})$ has a positive maximum on ω_h at $\overline{\boldsymbol{x}}$, i.e., $u(\overline{\boldsymbol{x}}) = \max_{\boldsymbol{x}\in\omega_h} u(\boldsymbol{x}) > 0$.

For the node $\overline{\boldsymbol{x}}$, using (2.8), it holds that

$$L_h u(\overline{\boldsymbol{x}}) = \underbrace{d(\overline{\boldsymbol{x}})}_{\ge 0}\underbrace{u(\overline{\boldsymbol{x}})}_{>0} - \sum_{\boldsymbol{y}\in S(\overline{\boldsymbol{x}})} \underbrace{b(\overline{\boldsymbol{x}},\boldsymbol{y})}_{>0} \underbrace{\big(u(\boldsymbol{y}) - u(\overline{\boldsymbol{x}})\big)}_{\le 0 \text{ by definition of } \overline{\boldsymbol{x}}} \ge d(\overline{\boldsymbol{x}})u(\overline{\boldsymbol{x}}) \ge 0. \tag{2.14}$$

Hence, it follows that $L_h u(\overline{\boldsymbol{x}}) = 0$ and, in particular, that all terms of $L_h u(\overline{\boldsymbol{x}})$ have to vanish. For the first term, it follows that $d(\overline{\boldsymbol{x}}) = 0$. For the terms in the sum to vanish, it must hold

$$u(\boldsymbol{y}) = u(\overline{\boldsymbol{x}}) \quad \forall\, \boldsymbol{y} \in S(\overline{\boldsymbol{x}}). \tag{2.15}$$

From the assumption $u(\boldsymbol{x}) \neq const$, it follows that there exists a node $\hat{\boldsymbol{x}} \in \omega_h$ with $u(\overline{\boldsymbol{x}}) > u(\hat{\boldsymbol{x}})$. Because the grid is connected, there is a path $\overline{\boldsymbol{x}}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, \hat{\boldsymbol{x}}$ in ω_h such that, using (2.15) for all nodes of this path,

$$\begin{aligned}
&\boldsymbol{x}_1 \in S(\overline{\boldsymbol{x}}),\ \ u(\boldsymbol{x}_1) = u(\overline{\boldsymbol{x}}),\\
&\boldsymbol{x}_2 \in S(\boldsymbol{x}_1),\ u(\boldsymbol{x}_2) = u(\boldsymbol{x}_1) = u(\overline{\boldsymbol{x}}),\\
&\ldots\\
&\hat{\boldsymbol{x}} \in S(\boldsymbol{x}_m),\ u(\boldsymbol{x}_m) = u(\boldsymbol{x}_{m-1}) = \ldots = u(\overline{\boldsymbol{x}}) > u(\hat{\boldsymbol{x}}).
\end{aligned}$$

The last inequality is a contradiction to (2.15) for $\boldsymbol{x}_m$. ■

Remark 2.20. On L_h. Note that L_h is defined for the inner nodes, i.e., this operator corresponds to the rectangular matrix (A_1, A_2) from (2.9). □

Corollary 2.21. DMP for the finite difference boundary value problem. *Let $u(\boldsymbol{x}) \le 0$ for $\boldsymbol{x} \in \gamma_h$ and $L_h u(\boldsymbol{x}) \le 0$ (or $u(\boldsymbol{x}) \ge 0$ for $\boldsymbol{x} \in \gamma_h$ and $L_h u(\boldsymbol{x}) \ge 0$, respectively) on ω_h. Assume that there is at least one inner node close to the boundary $\boldsymbol{x}^*$ and one node $\boldsymbol{x}_\gamma$ on the boundary with*

$b(\boldsymbol{x}^*, \boldsymbol{x}_\gamma) > 0$, i.e., the matrix block A_2 in (2.9) is not the zero matrix. Then, the grid function $u(\boldsymbol{x})$ is non-positive (or non-negative, respectively) for all $\boldsymbol{x} \in \omega_h \cup \gamma_h$.

Proof. Let $L_h u(\boldsymbol{x}) \le 0$ on ω_h. Assume that there is a node $\overline{\boldsymbol{x}} \in \omega_h$ with $u(\overline{\boldsymbol{x}}) > 0$. Then, the grid function has either a positive maximum on ω_h and it is not constant, which is a contradiction to the DMP for the inner nodes, Lemma 2.19, or $u(\boldsymbol{x})$ has to be constant, i.e., $u(\boldsymbol{x}) = u(\overline{\boldsymbol{x}}) > 0$ for all $\boldsymbol{x} \in \omega_h$. For the second case, consider the boundary-connected inner node $\boldsymbol{x}^* \in \omega_h^*$. Using the same calculations as in (2.14) and taking into account that the values of u at the boundary are non-positive, one obtains

$$\begin{aligned} L_h u(\boldsymbol{x}^*) &= \underbrace{d(\boldsymbol{x}^*)}_{\ge 0}\underbrace{u(\boldsymbol{x}^*)}_{>0} - \sum_{\boldsymbol{y}\in S(\boldsymbol{x}^*), \boldsymbol{y}\notin\gamma_h} \underbrace{b(\boldsymbol{x}^*, \boldsymbol{y})}_{>0}\underbrace{(u(\boldsymbol{y}) - u(\boldsymbol{x}^*))}_{=0} \\ &\quad - \sum_{\boldsymbol{y}\in S(\boldsymbol{x}^*), \boldsymbol{y}\in\gamma_h} \underbrace{b(\boldsymbol{x}^*, \boldsymbol{y})}_{>0}\underbrace{(u(\boldsymbol{y}) - u(\boldsymbol{x}^*))}_{<0} > 0. \end{aligned} \tag{2.16}$$

In the last sum, there is at least one term since $\boldsymbol{x}_\gamma \in S(\boldsymbol{x}^*)$. Altogether, (2.16) is a contradiction to the assumption on L_h. ■

Corollary 2.22. Unique solution of the discrete Laplace equation with homogeneous right-hand side and homogeneous Dirichlet boundary conditions. *Under the assumptions of Corollary 2.21, the discrete Laplace equation $L_h u(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \omega_h$ and $u(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \gamma_h$ possesses only the trivial solution $u(\boldsymbol{x}) = 0$.*

Proof. The statement of the corollary follows by applying Corollary 2.21 both for $L_h u(\boldsymbol{x}) \le 0$ and $L_h u(\boldsymbol{x}) \ge 0$. ■

Theorem 2.23. Existence and uniqueness of a solution of the finite difference equation (2.6). *Under the assumptions of Corollary 2.22, the finite difference equation (2.6) possesses a unique solution.*

Proof. Corollary 2.22 shows that the homogeneous linear system of equations (2.9) has a unique solution. Hence, the system matrix is invertible and it follows that (2.9) is uniquely solvable for all right-hand sides, where (2.9) is just the matrix-vector representation of (2.6). ■

Corollary 2.24. Comparison lemma. *Let the assumptions of Corollary 2.21 be satisfied and let*

$$\begin{aligned} L_h u(\boldsymbol{x}) &= f(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \omega_h; \quad u(\boldsymbol{x}) = g(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \gamma_h, \\ L_h \overline{u}(\boldsymbol{x}) &= \overline{f}(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \omega_h; \quad \overline{u}(\boldsymbol{x}) = \overline{g}(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \gamma_h, \end{aligned}$$

with $|f(\boldsymbol{x})| \le \overline{f}(\boldsymbol{x})$, $\boldsymbol{x} \in \omega_h$, and $|g(\boldsymbol{x})| \le \overline{g}(\boldsymbol{x})$, $\boldsymbol{x} \in \gamma_h$. Then, it is $|u(\boldsymbol{x})| \le \overline{u}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \omega_h \cup \gamma_h$. The function $\overline{u}(\boldsymbol{x})$ is called majorizing function.

Proof. Exercise. ■

Remark 2.25. Remainder of this section. The remaining corollaries presented in this section will be applied in the stability proof in Section 2.4. In this