Chapter 7
Finite Element Methods for Second Order Elliptic Problems

7.1 General Convergence Theorems

Remark 7.1. Motivation. There are many situations that are not covered by the Ritz or Galerkin theory from Section 4.3 and for which an extension of these theories are needed.

In Section 5.1, non-conforming finite element spaces were introduced, i.e., methods where the finite element space V^h is not a subspace of V, which is the space in the definition of the continuous variational problem. The property $V^h \not\subset V$ is given for the Crouzeix–Raviart and the Rannacher–Turek element. Another case of non-conformity is given if the domain does not possess a polyhedral boundary and one has to apply some approximation of the boundary.

For non-conforming methods, the finite element approach is not longer a Ritz method. Hence, the convergence proof from Theorem 4.14 cannot be applied in this case. In addition, in practice, one is interested also in the order of convergence in other norms than $\|\cdot\|_V$ or one has to take into account that the values of the bilinear or linear form need to be approximated numerically. The abstract convergence theorem, which will be proved in this section, allows the numerical analysis of complex finite element methods. □

Remark 7.2. Notations, Assumptions. Let $\{h > 0\}$ be a set of mesh widths and let S^h, V^h normed spaces of functions which are defined on domains $\{\Omega^h \subset \mathbb{R}^d\}$. It will be assumed that the space S^h has a finite dimension and that S^h and V^h possess a common norm $\|\cdot\|_h$. In the application of the abstract theory, S^h will be a finite element space and V^h is defined so that the restriction and/or extension of the solution of the continuous problem to Ω^h is contained in V^h. The index h indicates that V^h might depend on h but not that V^h is finite-dimensional. Strictly speaking, the modified solution of the continuous problem does not solve the given problem any longer. Hence, it is natural that the continuous problem does not appear explicitly in the abstract theory.

Given the bilinear forms

$$a^h : S^h \times S^h \to \mathbb{R},$$
$$\tilde{a}^h : (S^h + V^h) \times (S^h + V^h) \to \mathbb{R}.$$

Let the bilinear form a^h be regular in the sense that there is a constant $m > 0$, which is independent of h, such that for each $v^h \in S^h$ there is a $w^h \in S^h$ with $\left\|w^h\right\|_h = 1$ so that[1]

$$m\left\|v^h\right\|_h \le a^h(v^h, w^h). \tag{7.1}$$

This condition is equivalent to the requirement that the stiffness matrix A with the entries $a_{ij} = a^h(\phi_j, \phi_i)$, where $\{\phi_i\}$ is a basis of S^h, is uniformly non-singular, i.e., its non-singularity is independent of h (eigenvalues are bounded away from zero uniformly with respect to h). For the second bilinear form, only its boundedness will be assumed

$$\tilde{a}^h(u, v) \le M\left\|u\right\|_h \left\|v\right\|_h \quad \forall\ u, v \in S^h + V^h. \tag{7.2}$$

Let the linear functionals $\{f^h(\cdot)\}\ :\ S^h \to \mathbb{R}$ be given. Then, the following discrete problems will be considered: Find $u^h \in S^h$ with

$$a^h(u^h, v^h) = f^h(v^h) \quad \forall\ v^h \in S^h. \tag{7.3}$$

Because the stiffness matrix is assumed to be non-singular, there is a unique solution of (7.3).

Note the similarities of the whole setup with the assumptions for the Theorem of Lax–Milgram. In fact, the current setup can be considered as a generalization of the Lax–Milgram theory. □

Theorem 7.3. Abstract finite element error estimate. *Let the conditions* (7.1) *and* (7.2) *be satisfied and let* u^h *be the solution of* (7.3)*. Then, the following error estimate holds for each* $\tilde{u} \in V^h$

$$\begin{aligned}\left\|\tilde{u} - u^h\right\|_h \le C \inf_{v^h \in S^h}\left\{\left\|\tilde{u} - v^h\right\|_h + \sup_{w^h \in S^h}\frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\left\|w^h\right\|_h}\right\}\\ +C\sup_{w^h \in S^h}\frac{\left|\tilde{a}^h(\tilde{u}, w^h) - f^h(w^h)\right|}{\left\|w^h\right\|_h}\end{aligned} \tag{7.4}$$

with $C = C(m, M)$*.*

[1] note that this condition can be formulated as an inf-sup condition:

$$0 < m \le \inf_{v^h \in S^h}\sup_{w^h \in S^h}\frac{a^h(v^h, w^h)}{\left\|v^h\right\|_h\left\|w^h\right\|_h}$$

Proof. Because of (7.1), there is for each $v^h \in S^h$ a $w^h \in S^h$ with $\left\|w^h\right\|_h = 1$ and

$$m\left\|u^h - v^h\right\|_h \le a^h(u^h - v^h, w^h).$$

Using the definition of u^h from (7.3), one obtains

$$m\left\|u^h - v^h\right\|_h \le f^h(w^h) - a^h(v^h, w^h) + \tilde{a}^h(v^h, w^h) + \tilde{a}^h(\tilde{u} - v^h, w^h) - \tilde{a}^h(\tilde{u}, w^h).$$

From (7.2) and $\left\|w^h\right\|_h = 1$, it follows that

$$\tilde{a}^h(\tilde{u} - v^h, w^h) \le M\left\|\tilde{u} - v^h\right\|_h.$$

Rearranging the terms appropriately and using $\left\|w^h / \left\|w^h\right\|_h\right\|_h = 1$ yields

$$\begin{aligned} m\left\|u^h - v^h\right\|_h \le M\left\|\tilde{u} - v^h\right\|_h &+ \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\left\|w^h\right\|_h} \\ &+ \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(\tilde{u}, w^h) - f^h(w^h)\right|}{\left\|w^h\right\|_h}. \end{aligned} \tag{7.5}$$

Applying the triangle inequality

$$\left\|\tilde{u} - u^h\right\|_h \le \left\|\tilde{u} - v^h\right\|_h + \left\|u^h - v^h\right\|_h,$$

inserting the estimate (7.5), and taking into account that v^h was chosen arbitrarily, so that the infimum can be taken, gives (7.4). ■

Remark 7.4. To Theorem 7.3.

- An important special case of this theorem is the case that the stiffness matrix is uniformly positive definite, i.e., the condition

$$m\left\|v^h\right\|_h^2 \le a^h(v^h, v^h) \quad \forall\, v^h \in S^h,\ \forall\, h. \tag{7.6}$$

  is satisfied. Dividing (7.6) by $\left\|v^h\right\|_h$ reveals that condition (7.1) is implied by (7.6).
- If the continuous problem is also defined with the bilinear form $\tilde{a}^h(\cdot,\cdot)$, then

$$\sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\left\|w^h\right\|_h}$$

  can be considered as consistency error of the bilinear forms, i.e., it measures the difference between the bilinear forms used in the continuous and discrete problem, and the term

$$\sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(\tilde{u}, w^h) - f^h(w^h)\right|}{\left\|w^h\right\|_h}$$

  as consistency error of the right-hand sides.

□

Theorem 7.5. First Strang[2] lemma *Let S^h be a conforming finite element space, i.e., $S^h \subset V$, with $\|\cdot\|_h = \|\cdot\|_V$ and let the space V^h be independent of h. Consider a continuous problem of the form*

$$\tilde{a}^h(u,v) = f(v) \quad \forall\, v \in V,$$

then the following error estimate holds

$$\begin{aligned}\left\|u-u^h\right\|_V \le C \inf_{v^h\in S^h}\left\{\left\|u-v^h\right\|_V + \sup_{w^h\in S^h}\frac{\left|\tilde{a}^h(v^h,w^h)-a^h(v^h,w^h)\right|}{\|w^h\|_V}\right\}\\ +C\sup_{w^h\in S^h}\frac{\left|f(w^h)-f^h(w^h)\right|}{\|w^h\|_V}.\end{aligned}$$

Proof. The statement of this theorem follows directly from Theorem 7.3. ■

7.2 Finite Element Method with the Non-Conforming Crouzeix–Raviart Element

Remark 7.6. The continuous problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz boundary. Let

$$Lu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{7.7}$$

where the operator is given by

$$Lu = -\nabla\cdot(A\nabla u)$$

with $A = A^T$ and

$$A(\boldsymbol{x}) = (a_{ij}(\boldsymbol{x}))_{i,j=1}^d, \quad a_{ij} \in W^{1,p}(\Omega), p > d. \tag{7.8}$$

It will be assumed that there are two positive real numbers m, M such that

$$m\,\|\boldsymbol{\xi}\|_2^2 \le \boldsymbol{\xi}^T A(\boldsymbol{x})\boldsymbol{\xi} \le M\,\|\boldsymbol{\xi}\|_2^2 \quad \forall\, \boldsymbol{\xi}\in\mathbb{R}^d, \boldsymbol{x}\in\overline{\Omega}. \tag{7.9}$$

Hence, $A(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x} \in \overline{\Omega}$, so that the operator L is elliptic, see Definition 1.18. From the Sobolev inequality, Theorem 3.51, it follows that $a_{ij} \in L^\infty(\Omega)$. With

$$a(u,v) = \int_\Omega (A(\boldsymbol{x})\nabla u(\boldsymbol{x}))\cdot\nabla v(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}$$

and the Cauchy–Schwarz inequality, one obtains

[2] Gilbert Strang, born 1934

$$|a(u,v)| \le \|A\|_{L^\infty(\Omega)} \int_\Omega |\nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})| \ \mathrm{d}\boldsymbol{x} \le C \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

for all $u, v \in H_0^1(\Omega)$. In addition, it follows from (7.9) that

$$m \|\nabla u\|_{L^2(\Omega)}^2 \le a(u,u) \quad \forall\, u \in H_0^1(\Omega).$$

Hence, the bilinear form is bounded and elliptic. Using the Theorem of Lax–Milgram, Theorem 4.5, it follows that for given $f \in H^{-1}(\Omega)$ there es a unique weak solution $u \in H_0^1(\Omega)$ of

$$a(u,v) = f(v) \quad \forall\, v \in H_0^1(\Omega). \tag{7.10}$$

□

Remark 7.7. Assumptions and the discrete problem. The non-conforming Crouzeix–Raviart finite element P_1^{nc} was introduced in Example 5.30. To simplify the presentation, it will be restricted here on the two-dimensional case. In addition, to avoid the estimate of the error coming from approximating the domain, it will be assumed that Ω is a convex domain with polygonal boundary. It can be shown that in this case the boundary is Lipschitz. In addition, it is assumed that $f \in L^2(\Omega)$ and $a_{ij} \in W^{1,\infty}(\Omega)$.

Let $\{\mathcal{T}^h\}$ be a family of regular triangulations of Ω with triangles. Let P_1^{nc} (nc – non-conforming) denote the finite element space of piecewise linear functions that are continuous at the midpoints of the edges. This space is non-conforming if it is applied for the discretization of a second order elliptic equation since the continuous problem is given in $H_0^1(\Omega)$ and the functions of $H_0^1(\Omega)$ do not possess jumps. The functions of P_1^{nc} have generally jumps, see Figure 7.1, and they are not weakly differentiable. In addition, the space is also non-conforming with respect to the boundary condition, which is not satisfied exactly. The functions from P_1^{nc} that will be sought as an approximation of the solution of the boundary value problem (7.7) vanish in the midpoint of the edges at the boundary. However, in the other points at the boundary, their value is generally not equal to zero.

The bilinear form

$$a(u,v) = \int_\Omega (A(\boldsymbol{x})\nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x}$$

will be extended to $H_0^1(\Omega) + P_1^{\mathrm{nc}}$ by

$$a^h(u,v) = \sum_{K \in \mathcal{T}^h} \int_K (A(\boldsymbol{x})\nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x} \quad \forall\, u, v \in H_0^1(\Omega) + P_1^{\mathrm{nc}}.$$

Then, the non-conforming finite element method is given by: Find $u^h \in P_1^{\mathrm{nc}}$ with

$$a^h(u^h, v^h) = (f, v^h) \quad \forall\, v^h \in P_1^{\mathrm{nc}}.$$

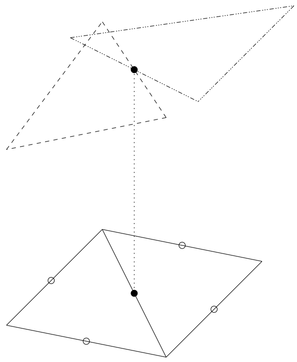

Fig. 7.1 Function from P_1^{nc}.

The goal of this section consists in proving the linear convergence with respect to h in the energy norm $\|\cdot\|_h = \left(a^h(\cdot,\cdot)\right)^{1/2}$. It can be proved that the solution of the continuous problem (7.10) is smooth, i.e., that $u \in H^2(\Omega)$, since $f \in L^2(\Omega)$, the coefficients $a_{ij}(\boldsymbol{x})$ are weakly differentiable with bounded derivatives, and Ω is a convex domain with polygonal boundary. □

Remark 7.8. The error equation. The first step of proving an error estimate consists in deriving an equation for the error. To this end, multiply the continuous problem (7.7) with a test function from $v^h \in P_1^{\mathrm{nc}}$, integrate the product on Ω, and apply integration by parts on each triangle. This approach gives

$$\begin{aligned}(f, v^h) &= -\sum_{K\in\mathcal{T}^h}\int_K \nabla\cdot\left(A(\boldsymbol{x})\nabla u(\boldsymbol{x})\right)v^h(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}\\ &= \sum_{K\in\mathcal{T}^h}\int_K \left(A(\boldsymbol{x})\nabla u(\boldsymbol{x})\right)\cdot\nabla v^h(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}\\ &\quad - \sum_{K\in\mathcal{T}^h}\int_{\partial K}\left(A(s)\nabla u(s)\right)\cdot\boldsymbol{n}_K(s)v^h(s)\ \mathrm{d}s\\ &= a^h(u, v^h) - \sum_{K\in\mathcal{T}^h}\int_{\partial K}\left(A(s)\nabla u(s)\right)\cdot\boldsymbol{n}_K(s)v^h(s)\ \mathrm{d}s,\end{aligned}$$

where $\boldsymbol{n}_K$ is the unit outer normal at the edges of the triangles. Subtracting the finite element equation, one obtains

$$a^h(u-u^h, v^h) = -\sum_{K\in\mathcal{T}^h}\int_{\partial K}\left(A(s)\nabla u(s)\right)\cdot\boldsymbol{n}_K(s)v^h(s)\ \mathrm{d}s \quad \forall\ v^h \in P_1^{\mathrm{nc}}. \tag{7.11}$$

□

Lemma 7.9. Estimate of the right-hand side of the error equation (7.11)**.** *Assume that $u \in H^2(\Omega)$ and $a_{ij} \in W^{1,\infty}(\Omega)$, $i,j = 1,2$, then it is*