Chapter 3
Introduction to Sobolev Spaces

Remark 3.1. Contents. Sobolev spaces are the basis of the theory of weak or variational forms of partial differential equations. A very popular approach for discretizing partial differential equations, the finite element method, is based on variational forms. In this chapter, a short introduction into Sobolev spaces will be given. Recommended literature are the books Adams (1975); Adams & Fournier (2003), and Evans (2010). □

3.1 Elementary Inequalities

Lemma 3.2. Inequality for strictly monotonically increasing function. *Let $f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ be a continuous and strictly monotonically increasing function with $f(0) = 0$ and $f(x) \to \infty$ for $x \to \infty$. Then, for all $a, b \in \mathbb{R}_+ \cup \{0\}$, it is*

$$ab \le \int_0^a f(x)\ dx + \int_0^b f^{-1}(y)\ dy,$$

where $f^{-1}(y)$ is the inverse of $f(x)$.

Proof. Since $f(x)$ is strictly monotonically increasing, the inverse function exists.

The proof is based on a geometric argument, see Figure 3.1.

Consider the interval $(0, a)$ on the x-axis and the interval $(0, b)$ on the y-axis. Then, the area of the corresponding rectangle is given by ab, $\int_0^a f(x)\ dx$ is the area below the curve, and $\int_0^b f^{-1}(y)\ dy$ is the area between the positive y-axis and the curve. From Figure 3.1, the inequality follows immediately. The equal sign holds only iff $f(a) = b$. ■

Remark 3.3. Young's[1] inequality. Young's inequality

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \quad \forall\ a, b \in \mathbb{R}_+ \cup \{0\}, \varepsilon \in \mathbb{R}_+, \tag{3.1}$$

[1] William Henry Young (1863 – 1942)

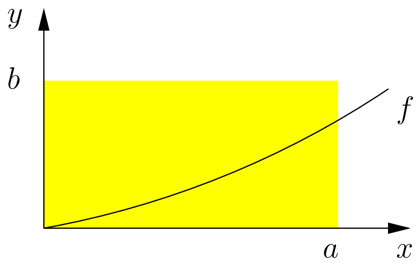

Fig. 3.1 Sketch to the proof of Lemma 3.2.

follows from Lemma 3.2 with $f(x) = \varepsilon x$, $f^{-1}(y) = \varepsilon^{-1} y$. It is also possible to derive this inequality from the binomial theorem. For proving the generalized Young inequality

$$ab \le \frac{\varepsilon^p}{p} a^p + \frac{1}{q\varepsilon^q} b^q, \quad \forall\ a, b \in \mathbb{R}_+ \cup \{0\}, \varepsilon \in \mathbb{R}_+, \tag{3.2}$$

with $p^{-1}+q^{-1} = 1, p, q \in (1,\infty)$, one chooses $f(x) = x^{p-1}$, $f^{-1}(y) = y^{1/(p-1)}$ and applies Lemma 3.2 with intervals where the upper bounds are given by εa and $\varepsilon^{-1} b$. □

Remark 3.4. Cauchy–Schwarz inequality.

- The Cauchy[2]–Schwarz[3] inequality (for vectors, for sums)

$$\left|(\underline{x}, \underline{y})\right| \le \|\underline{x}\|_2 \left\|\underline{y}\right\|_2 \ \forall\ \underline{x}, \underline{y} \in \mathbb{R}^n, \tag{3.3}$$

  where $(\cdot,\cdot)$ is the Euclidean inner product and $\|\cdot\|_2$ the Euclidean norm, is well known.
- One can prove this inequality with the help of Young's inequality. First, it is clear that the Cauchy–Schwarz inequality is correct if one of the vectors is the zero vector. Now, let $\underline{x}, \underline{y}$ with $\|\underline{x}\|_2 = \left\|\underline{y}\right\|_2 = 1$. One obtains with the triangle inequality and Young's inequality (3.1)

$$\left|(\underline{x}, \underline{y})\right| = \left|\sum_{i=1}^n x_i y_i\right| \le \sum_{i=1}^n |x_i|\,|y_i| \le \frac{1}{2}\sum_{i=1}^n |x_i|^2 + \frac{1}{2}\sum_{i=1}^n |y_i|^2 = 1.$$

  Hence, the Cauchy–Schwarz inequality is correct for $\underline{x}, \underline{y}$. Last, one considers arbitrary vectors $\underline{\tilde{x}} \ne \underline{0}, \underline{\tilde{y}} \ne \underline{0}$. Now, one can utilize a homogeneity argument. From the validity of the Cauchy–Schwarz inequality for $\underline{x}$ and $\underline{y}$, one obtains by a scaling argument

[2] Augustin Louis Cauchy (1789 – 1857)

[3] Hermann Amandus Schwarz (1843 – 1921)

$$\left|(\underbrace{\|\underline{\tilde{x}}\|_2^{-1}\,\underline{\tilde{x}}}_{\underline{x}}, \underbrace{\|\underline{\tilde{y}}\|_2^{-1}\,\underline{\tilde{y}}}_{\underline{y}})\right| \le 1.$$

Both vectors $\underline{x}, \underline{y}$ have the Euclidean norm 1, hence

$$\frac{1}{\|\underline{\tilde{x}}\|_2 \|\underline{\tilde{y}}\|_2} \left|(\underline{\tilde{x}}, \underline{\tilde{y}})\right| \le 1 \quad \Longleftrightarrow \quad \left|(\underline{\tilde{x}}, \underline{\tilde{y}})\right| \le \|\underline{\tilde{x}}\|_2 \|\underline{\tilde{y}}\|_2 .$$

- The generalized Cauchy–Schwarz inequality or Hölder's[4] inequality

$$\left|(\underline{x}, \underline{y})\right| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

with $p^{-1} + q^{-1} = 1, p, q \in (1, \infty)$, can be proved in the same way with the help of the generalized Young inequality.

□

Definition 3.5. *Lebesgue spaces.* The space of functions that are Lebesgue[5] integrable on Ω to the power of $p \in [1, \infty)$ is denoted by

$$L^p(\Omega) = \left\{ f \ : \ \int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x} < \infty \right\},$$

which is equipped with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x} \right)^{1/p}.$$

For $p = \infty$, this space is given by

$$L^\infty(\Omega) = \{ f \ : \ |f(\boldsymbol{x})| < \infty \text{ almost everywhere in } \Omega \}$$

with the norm

$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{\boldsymbol{x} \in \Omega} |f(\boldsymbol{x})|.$$

□

Lemma 3.6. Hölder's inequality. *Let $p^{-1} + q^{-1} = 1, p, q \in [1, \infty]$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then it is $uv \in L^1(\Omega)$ and it holds that*

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} . \tag{3.4}$$

If $p = q = 2$, then this inequality is also known as Cauchy–Schwarz inequality

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[4] Otto Hölder (1859 – 1937)

[5] Henri Lebesgue (1875 – 1941)