Department for Mathematics and Computer Science
Free University of Berlin
Prof. Dr. V. John, john@wias-berlin.de
Marwa Zainelabdeen, marwazinabdeen@gmail.com

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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 06

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Poincaré–Friedrichs type inequality.* Prove the following inequality of Poincaré–Friedrichs type. Let Ω be a bounded domain with Lipschitz boundary and let $\Omega' \subset \Omega$ with $\mathrm{meas}_{\mathbb{R}^d}(\Omega') = \int_{\Omega'} \, d\mathbf{x} > 0$, then for all $u \in W^{1,p}(\Omega)$, $p \in [1, \infty)$, it is
$$\int_\Omega |u(\mathbf{x})|^p \ d\mathbf{x} \le C \left(\left| \int_{\Omega'} u(\mathbf{x}) \ d\mathbf{x} \right|^p + \int_\Omega \|\nabla u(\mathbf{x})\|_2^p \ d\mathbf{x} \right),$$
where $\|\cdot\|_2$ denotes the Euclidean norm of a vector, which is different than the l^p vector norm for $p \neq 2$. **4 points**

2. *Integration by parts.*

   (a) Prove Corollary 3.44: Let the conditions of Theorem 3.42 on the domain Ω be satisfied. Consider $w \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, it is
$$\int_\Omega \partial_i w(\mathbf{x}) v(\mathbf{x}) \ d\mathbf{x} = \int_\Gamma w(\boldsymbol{s}) v(\boldsymbol{s}) \boldsymbol{n}_i(\boldsymbol{s}) \ d\boldsymbol{s} - \int_\Omega w(\mathbf{x}) \partial_i v(\mathbf{x}) \ d\mathbf{x}.$$
2 points

   (b) Let Ω be a bounded Lipschitz domain and let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Prove the interpolation inequality
$$\|\nabla u\|_{L^2(\Omega)}^2 \le \|u\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)},$$
with
$$\|\nabla u\|_{L^2(\Omega)} = \left(\int_\Omega \nabla u \cdot \nabla u \ d\mathbf{x} \right)^{1/2}.$$
2 points

3. *Imbedding of Sobolev spaces with the same order of the derivative k and different integration powers.* Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $k \ge 0$, and $p, q \in [1, \infty]$ with $q > p$. Then, it is $W^{k,q}(\Omega) \subset W^{k,p}(\Omega)$. **3 points**

4. *Alternative inner product in $L^2(0, \infty)$.* Show that
$$a(u, v) = \int_0^\infty e^{-x} u(x) v(x) \ dx$$
defines a (real) inner product in $L^2(0, \infty)$.
Hint: A map $a(\cdot, \cdot) \ : \ V \times V \to \mathbb{R}$ is a (real) inner product if it is bilinear, symmetric, and positive definite, i.e.:

i) $a(\alpha u+\beta v, w) = \alpha a(u,v)+\beta a(w,v),\ a(u, \alpha v+\beta w) = \alpha a(u,v)+\beta a(u,w),\ \forall\, u,v,w \in V, \alpha, \beta \in \mathbb{R}$,

ii) $a(u,v) = a(v,u)\ \forall\, u,v \in V$,

iii) $a(u,u) \geq 0\ \forall\, u \in V$ and $a(u,u) = 0 \iff u = 0$.

4 points

The exercise problems should be solved in groups of four to five students. The solutions have to be submitted until **Thursday, June 12th, 2023, 10:00 a.m.** via the whiteboard.