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Numerik I

English translation of Übungsserie 12 (final exercise sheet)

Attention: Only solutions which provide a comprehensible reasoning will be graded. Every statement has to be argued. You can use results from the lecture. Statments without reasoning won't get any points.

1. *Forward Euler method.* Consider the initial value problem

$$(1+x)y'(x) + y(x) = \frac{1}{1+x}, \quad y(0) = 1.$$

   (a) Compute an approximation of the solution with the forward Euler method in $[0,1]$ with the step lengths $h_1 = 0.2$ and $h_2 = 0.1$.
   Hint: write a short code.

   (b) Compute the error to the analytical solution

$$y(x) = \frac{\ln(x+1)+1}{1+x}$$

   at $x = 1$.

   (c) Discuss the results briefly.

4 points

2. *Estimate for a sequence of real numbers.* Assume that for real numbers x_n, $n = 0, 1, \ldots$, the inequality

$$|x_{n+1}| \le (1+\delta)\,|x_n| + \beta$$

   holds with constants $\delta > 0$, $\beta \ge 0$. Then, it holds that

$$|x_n| \le e^{n\delta}\,|x_0| + \frac{e^{n\delta}-1}{\delta}\beta, \quad n = 0, 1, \ldots.$$

3 points

3. *Consistency conditions for a 3^{rd} order Runge–Kutta scheme with three stages.* Consider an autonomous initial value problem

$$y'(x) = f(y(x)), \quad y(x_0) = y_0.$$

   This problem shall be solved on an equidistant grid with an explicit 3-stage Runge–Kutta method. Derive the conditions for this method of being of third order that are given in the course, where for the coefficients of the Runge–Kutta scheme it shall hold

$$c_2 = a_{21}, \quad c_3 = a_{31} + a_{32}.$$

   Hint: Use the same approach as for the second order method which was presented in the course.

5 points

4. *Finite Difference matrix for the second order derivative.* Consider the differential equation (Poisson equation, boundary value problem)

$$\begin{aligned} -u'' &= f \quad \text{in } (0,1), \\ u(0) &= a, \\ u(1) &= b. \end{aligned}$$

Use an equidistant grid

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1, \quad h = x_i - x_{i-1},\ i = 1, \ldots, n,$$

for the discretization of the second derivative.

(a) Show that the approximation (finite difference)

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} =: u_{\overline{x}x,i}, \quad i = 1, \ldots, n-1$$

is of second order consistent, i.e.,

$$u_{\overline{x}x,i} = u''(x_i) + \mathcal{O}(h^2)$$

if $u \in C^4([0,1])$.

(b) Insert the approximation of the second order derivative in the differential equation and derive a linear system of equations for the values $u_i = u(x_i)$, $i = 0, \ldots, n$. Set the boundary conditions and use as approximation for the right-hand side $f_i = f(x_i)$, $i = 1, \ldots, n-1$.

(c) Reduce this linear system of equations to a system for u_i, $i = 1, \ldots, n-1$, by eliminating the equations for the boundary conditions.

3 points

The exercises should be solved in groups of two students. They have to be submitted until Sie **Wednesday, 04.02.2026, 10:00** electronically via whiteboard.