Fachbereich Mathematik und Informatik/ Institut für Mathematik
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Numerik I

English translation of Übungsserie 09

Attention: Only solutions which provide a comprehensible reasoning will be graded. Every statement has to be argued. You can use results from the lecture. Statments without reasoning won't get any points.

1. *Absolute condition for the computation of roots.* Let $f \in C^n([a,b])$ and $x_0 \in (a,b)$ a root of
$$f(x) - \alpha = 0, \quad \alpha \in \mathbb{R},$$
with multiplicity n but not $(n+1)$.
   (a) Consider a small perturbation ε of the data α and prove the following estimate for the absolute condition K
   $$K \approx \left| \frac{n!}{f^{(n)}(x_0)} \right| \varepsilon^{1/n-1}.$$
   What does this imply for arbitrary small perturbations?
   (b) The computation of roots of polynomials is a special case of the above mentioned assignment. This is for example necessary if one want to compute the Eigenvalues of a matrix via the characteristic polynomial. Let $f(x)$ a polynomial of n-th degree and root x_0 with multiplicity n. How can you improve the statement of the first part.

   Hint: Use Taylor expansion of $f(x)$ at x_0 and approximate the term including the unknown intermediate point by the same term at x_0. **4 points**

2. *Gershgorin circles.* Estimate the Eigenvalues of
$$A = \begin{pmatrix} 2 & -1/2 & 0 & -1/2 \\ 0 & 4 & 0 & 2 \\ -1/2 & 0 & 6 & 1/2 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$
by the means of the Gershgorin circles. **2 points**

3. *Uniqueness of QR decomposition.* Let $A \in \mathbb{R}^{n \times n}$ be a matrix with full rank. Show that the QR decomposition $A = QR$ with an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n \times n}$ is unique, up to signs of rows and columns. **6 Punkte**

4. *Programming exercise: Power iteration.* Implement the Power iteration and approximate the Eigenvalue with largest absolute value of the following matrix
$$A(\alpha) = \begin{pmatrix} \alpha & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{pmatrix}$$

for $\alpha \in \{-30, 30\}$. As initial guess one is supposed to use the unit vector $\boldsymbol{e}_1$. The iteration should be stopped if one obtains the following bound on the relative decay of the iterate

$$\frac{|\lambda^{(k+1)} - \lambda^{(k)}|}{|\lambda^{(k+1)}|} < 10^{-10}.$$

How many iterations are necessary? Which statement from the lectures can explain the difference in the number of iterations. **4 points**

The exercises should be solved in groups of two students. They have to be submitted until Sie **Monday, 24.06.2024, 10:00**, either in the box of the tutor or electronically via whiteboard.