Fachbereich Mathematik und Informatik/ Institut für Mathematik
Prof. Dr. Volker John, john@wias-berlin.de
Balázs Kossovics, b.kossovics@fu-berlin.de

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Numerik I

English translation of Übungsserie 03

Attention: Only solutions which provide a comprehensible reasoning will be graded. Every statement has to be argued. You can use results from the lecture. Statments without reasoning won't get any points.

1. *Orthogonal matrices.* Let

$$\mathbb{O}_m(\mathbb{R}) := \left\{ Q \in \mathbb{R}^{m\times m} \ : \ Q^T = Q^{-1} \right\}$$

be the set of orthogonal matrices in $\mathbb{R}^{m\times m}$. Prove the following statements.

   i) Let $Q \in \mathbb{O}_m(\mathbb{R})$, then $Q^T \in \mathbb{O}_m(\mathbb{R})$.
   ii) Let $Q \in \mathbb{O}_m(\mathbb{R})$, then $|\det(Q)| = 1$.
   iii) Let $Q_1, Q_2 \in \mathbb{O}_m(\mathbb{R})$, then $Q_1Q_2 \in \mathbb{O}_m(\mathbb{R})$.
   iv) $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ is satisfied by every $\mathbf{x} \in \mathbb{R}^m$.
   v) For every matrix $A \in \mathbb{R}^{m\times m}$ we have $\|A\|_2 = \|QA\|_2 = \|AQ\|_2$.
   vi) For every matrix $A \in \mathbb{R}^{m\times m}$ we have $\kappa_2(A) = \kappa_2(QA) = \kappa_2(AQ)$.
   vii) Each $Q \in \mathbb{O}_m(\mathbb{R})$ satisfies $\kappa_2(Q) = 1$.

4 points

2. *Generalized inverse.* Let $A \in \mathbb{R}^{m\times n}$. The generalized inverse $A^+ \in \mathbb{R}^{n\times m}$ of A is uniquely determined by the Moore–Penrose conditions.

$$AA^+A = A,\ A^+AA^+ = A^+,\ (AA^+)^T = AA^+,\ (A^+A)^T = A^+A.$$

Compute the generalized inverse of $A = (1, 2, 3) \in \mathbb{R}^{1\times 3}$ utilizing these conditions. **2 points**

3. *Spectral condition number.* Let $A \in \mathbb{R}^{n\times n}$ a nonsingular matrix. Demonstrate the following identity for the condition number

$$\kappa_2(A^TA) = (\kappa_2(A))^2.$$

2 points

4. *Projection.* Let V be an inner product space and $P : V \to V$ a linear operator. Prove the following statements to be equivalent.

   (a) $(x - Px, y) = 0$ for every $x \in V$ and every $y \in \mathrm{im}(P) = \{Pz \ : \ z \in V\}$.
   (b) $P^2 = P$ and $(Px, y) = (x, Py)$ for every $x, y \in V$.

2 points

Do not forget the programming problem from Exercise Sheet 02!

The exercises should be solved in groups of two students. They have to be submitted until Sie **Monday, 06.05.2024, 10:00**, either in the box of the tutor or electronically via whiteboard.