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Numerical Mathematics II

Exercise Problems 03

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Repetition: Analytical solution of an initial value problem.* Solve the following initial value problem

$$xy'(x) + 3y(x) = x^2, \quad x_0 = 0,\ y_0 = 0.$$

2. *Failure of the explicit Euler method.* Consider the initial value problem

$$y'(t) = y(t)^{1/5}, \quad y(0) = 0.$$

   Determine the analytical solution of the given initial value problem. Show that the explicit Euler method fails to approximate the analytical solution.

3. *Matrices in classical iteration schemes.* Solve the following problems.

   (a) Let G_{GS} be the iteration matrix of the Gauss–Seidel method. Show that

$$G_{\mathrm{GS}} = -D^{-1}\left(LG_{\mathrm{GS}} + U\right). \tag{1}$$

   (b) Verify the following identities

$$D + \omega L = \left(1 - \frac{\omega}{2}\right) D + \frac{\omega}{2} A + \frac{\omega}{2}\left(L - U\right), \tag{2}$$

$$(1-\omega)\, D - \omega U = \left(1 - \frac{\omega}{2}\right) D - \frac{\omega}{2} A + \frac{\omega}{2}\left(L - U\right). \tag{3}$$

4. *Boundary value problem and the convergence of its finite difference approximation.* Consider the boundary value problem

$$\begin{aligned} -u'' &= f \quad \text{in } (0,1), \\ u(0) &= a, \\ u(1) &= b. \end{aligned}$$

   (a) Solve this problem analytically for

$$f(x) = -6\pi\cos(3\pi x) + 9\pi^2 x \sin(3\pi x),$$

   and $a = b = 0$.

   (b) Solve this problem numerically using the discretization described in Exercise Sheet 01, Problem 3, with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$. If you use MATLAB, you can use the backslash command. Give the error of the computed solution u_h to the analytic solution u in the following norm

$$\|u - u_h\|_{l^2} = \left(\frac{1}{N-1}\sum_{i=1}^{N-1}\left(u(x_i) - u_i\right)^2\right)^{1/2},$$

   where N is the number of nodes.

(c) Use the following ansatz of the convergence order

$$\|u - u_h\|_{l^2} = ch^\alpha.$$

Compute α by using the results on the two finest grids.

The exercise problems will be discussed at the turorial on **Thursday, May 07, 2026, 12-14**.