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Numerical Mathematics II

Exercise Problems 09

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Linear system of ordinary differential equations.* The restriction to scalar ordinary differential equations in most parts of the course is just for simplicity of presentation. All methods and most statements can be extended in a straightforward way to the vector-valued case, i.e., to systems of ordinary differential equations. In this problem, the analytic solution should be computed for one of the most simple cases for such systems.

   Find the general solution of the following linear system of ordinary differential equations

$$
\begin{aligned}
y'(x) &= x + y(x) + z(x) + u(x) \\
z'(x) &= x + z(x) + u(x) \\
u'(x) &= x + u(x) \, .
\end{aligned}
$$

   Hints:

   - Find first the solution of the homogeneous problem. To this end, insert equations in each other. Compute the other functions with a backward substitution.

   - Use an appropriate ansatz for finding a special solution of the inhomogeneous system.

   - Compare Appendix A, in particular A.1.3, if you are interested in the theory of linear systems of ordinary differential equations with constant coefficients.

   4 points

2. *Moore–Penrose inverse, pseudo inverse.* Let $A \in \mathbb{R}^{m \times n}$. The pseudo inverse or Moore–Penrose inverse $A^+ \in \mathbb{R}^{n \times m}$ of A is determined uniquely by the so-called Moore–Penrose conditions

$$
AA^+A = A, \ A^+AA^+ = A^+, \ (AA^+)^T = AA^+, \ (A^+A)^T = A^+A.
$$

   Compute on the basis of these conditions the pseudo inverse of $A = (2, 3, 4) \in \mathbb{R}^{1 \times 3}$. **4 points**

3. *Conjugate gradient (CG) method.* Continue Problem 3 from Exercise sheet 03. In this problem, iterative schemes had to be implemented for solving a linear system of equations with symmetric positive definite matrix. Consider again the meshes with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$.

(a) Implement the Conjugate Gradient method as given in Algorithm 6.8. How does the number of iteration change with h?

(b) If you use MATLAB, then apply the Conjugate Gradient method as provided by the routine `pcg`. Otherwise, find out if the used programming language provides a Conjugate Gradient method and use this method. Compare the number of iterations with your own implementation.

(c) Measure the executing times for both methods. If you use MATLAB, use the commands `tic` and `toc`. Which implementation is more efficient?

4 points

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, Jan. 06, 2025, 10:00 a.m.** via the whiteboard.