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Numerical Mathematics II

Exercise Problems 07

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Several methods applied to the model problem of linear stability.* Write a code that solves the model initial value problem for stability, problem (2.7), with the explicit Euler method, the implicit Euler method, and the trapezoidal rule. Take $\lambda = -10$, the interval $[0, 1]$, and meshes with 2, 4, 8, 16 intervals. Compute the error at the final point $x = 1$. Interprete the results.
   Hint: Use the formulas from Example 2.11 for implementing the methods. **6 points**

2. *Stability function of classical Runge–Kutta scheme.* Compute the stability function of the classical Runge–Kutta method and draw a sketch of the domain of stability.
   Hint: For drawing the domain of stability, one can compute sufficiently many values of the stability function, which can be performed quickly with a short program. The domain of stability is simply connected. **4 points**

3. *Second order boundary value problem with first order term and upwind discretization; GMRES.* Continue Problem 3 from Exercise Sheet 06.

   (a) Consider a decomposition of $[0, 1]$ by a grid as, e.g., in Problem 3, Exercise sheet 01. Show that the approximation (backward finite difference or upwind finite difference)
   $$u'(x_i) \approx \frac{u(x_i) - u(x_{i-1})}{h} = u_{\bar{x},i}, \quad i = 1, \ldots, n-1,$$
   is of first order, i.e.,
   $$u_{\bar{x},i} = u'(x_i) + \mathcal{O}(h)$$
   if $u \in C^2([0, 1])$. **1 point**

   (b) Modify the code of Problem 3 from Exercise Sheet 06 such that it applies to the differential equation given in this problem and such that the first order derivative is approximated by the backward difference. **2 points**

   (c) Consider the grids with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$ and compute the solution for $\varepsilon \in \{1, 10^{-2}, 10^{-4}, 10^{-6}\}$ (solve the linear system of equations with the backslash command) and compute the errors $\|u - u_h\|_{l^2}$. How does the error behaves with respect to the size ε ? **2 points**

   (d) Consider the same situations as in Problem 3c. Solve the arising linear systems of equations with the Jacobi method, the Gauss–Seidel method, and GMRES. In the used programming language, a provided GMRES routine can be used (MATLAB: read the documentation carefully). GMRES should be used without restart, with $tol = 1e-10$, and the maximal number of iterations should coincide with the dimension of the problem. Give the number of iterations for all methods ! How do they change with respect to ε and to h ? **2 points**

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, Dec. 02, 2024, 10:00 a.m.** via the whiteboard.