Department for Mathematics and Computer Science
Freie Universität Berlin
Prof. Dr. V. John, john@wias-berlin.de
André-Alexander Zepernick, a.zepernick@fu-berlin.de

Berlin, 18.11.2024

Numerical Mathematics II

Exercise Problems 06

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Radau-IA method.* A so-called Radau-IA method takes the left boundary of the interval as a node and it satisfies B$(2s-1)$ and D(s). Derive the Butcher tableau of the Radau-IA method for $s = 2$. **4 points**

2. *Representation of the numerical solution of the model IVP.* Prove the following theorem. Consider a Runge–Kutta method with s stages and with the parameters $(A, \mathbf{b}, \mathbf{c})$. If $z^{-1} = (\lambda h)^{-1}$ is not an eigenvalue of A, then the Runge–Kutta scheme is well-defined for the initial value problem (2.7). In this case, it is
$$y_k = (R(h\lambda))^k, \quad k = 0, 1, 2, \ldots.$$
2 points

3. *Second order boundary value problem with first order term.* Consider the following boundary value problem
$$-\varepsilon u''(x) + u'(x) = 1, \quad u(0) = u(1) = 0,$$
where $\varepsilon > 0$ is a parameter. The solution of this problem is
$$u(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$
   (a) Draw the solution in $[0, 1]$ for $\varepsilon \in \{1, 10^{-2}, 10^{-4}, 10^{-6}\}$. How does the solution change with respect to ε ? **2 points**

   (b) Consider a decomposition of $[0, 1]$ by a grid as, e.g., in Problem 3, Exercise sheet 01. Show that the approximation (central finite difference)
$$u'(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h} = u_{x,i}, \quad i = 1, \ldots, n-1,$$
   $x_{i-1} = x_i - h, x_{i+1} = x_i + h$, is of second order, i.e.,
$$u_{x,i} = u'(x_i) + \mathcal{O}(h^2)$$
   if $u \in C^3([0, 1])$. **1 point**

   (c) Modify the code of Problem 3, Exercise sheet 02, such that it applies to the differential equation given here, where the first order derivative is approximated by the central difference. **1 point**

   (d) Consider the grid with $h = 1/128$ and compute the solution for $\varepsilon \in \{1, 10^{-2}, 10^{-4}, 10^{-6}\}$ (solve the linear system of equations with the backslash command), compute the errors $\|u - u_h\|_{l^2}$, and draw the computed solutions. How do they change when ε becomes smaller ? **3 points**

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, Nov. 25, 2024, 10:00 a.m.** via the whiteboard.