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Numerical Mathematics II

Exercise Problems 03

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Consistency conditions for a 3rd order Runge–Kutta scheme with three stages.* Consider an autonomous initial value problem

$$y'(x) = f(y(x)), \quad y(x_0) = y_0.$$

This problem shall be solved on an equidistant grid with an explicit 3-stage Runge–Kutta method. Derive the conditions for this method of being of third order that are given in the course, where for the coefficients of the Runge–Kutta scheme it shall hold

$$c_2 = a_{21}, \quad c_3 = a_{31} + a_{32}.$$

Hint: Use the same approach as for the second order method which was presented in the course. **5 points**

2. *Matrices in classical iteration schemes.* Solve the following problems.

   (a) Let G_{GS} be the iteration matrix of the Gauss–Seidel method. Show that

$$G_{\mathrm{GS}} = -D^{-1}\left(LG_{\mathrm{GS}} + U\right). \tag{1}$$

   (b) Verify the following identities

$$\begin{aligned} D + \omega L &= \left(1 - \frac{\omega}{2}\right) D + \frac{\omega}{2} A + \frac{\omega}{2}\left(L - U\right), & (2) \\ \left(1 - \omega\right) D - \omega U &= \left(1 - \frac{\omega}{2}\right) D - \frac{\omega}{2} A + \frac{\omega}{2}\left(L - U\right). & (3) \end{aligned}$$

3 points

3. *Programming problem: damped Jacobi and SOR method.* Continue Problem 3 from Exercise Sheet 02. A sample code for Problem 3 from Exercise Sheet 02, also including an implementation of the Jacobi method, is available on the homepage of the course. If you do not have your own code, you can use this sample code as basis for solving the problems given below.

   (a) Compute the estimates of the spectral condition number for the matrices that are obtained with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$ with the MATLAB command `condest` or a corresponding command in other languages. What can be observed? **1 point**

(b) Implement the damped Jacobi method for solving the linear systems of equations that are obtained for $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$. Use the damping factors $\omega \in \{0.1, 0.2, \ldots, 1, 1.1\}$.

Use as starting iterate the zero vector and stop the iteration if the Euclidean norm of the residual $\|A\underline{u} - \underline{f}\|_2$ is less than 10^{-10} or after 100 000 iterations were performed. Count the number of iterations. What can be observed? **4 points**

(c) Implement the SOR method for solving the linear systems of equations that are obtained for $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$. Use the damping factors $\omega \in \{0.1, 0.2, \ldots, 1.8, 1.9, 2.0\}$.

Use as starting iterate the zero vector and stop the iteration if the Euclidean norm of the residual $\|A\underline{u} - \underline{f}\|_2$ is less than 10^{-10} or after 10 000 iterations were performed. Count the number of iterations. What can be observed? Give for each h the best relaxation factor (least number of iterations)! **4 points**

The simulations may take a while. The best way is to write a loop that performs one simulation after the other.

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, Nov. 04, 2024, 10:00 a.m.** via the whiteboard.