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Numerical Mathematics II

Exercise Problems 01

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Forward Euler method.* Consider the initial value problem

$$(1+x)y'(x) + y(x) = \frac{1}{1+x}, \quad y(0) = 1.$$

   (a) Compute an approximation of the solution with the forward Euler method in $[0,1]$ with the step lengths $h_1 = 0.2$ and $h_2 = 0.1$.
   Hint: write a short code.

   (b) Compute the error to the analytical solution

$$y(x) = \frac{\ln(x+1)+1}{1+x}$$

   at $x = 1$.

   (c) Discuss the results briefly.

4 points

2. *Vector norms.* Solve the following problems.

   (a) Let $\underline{x} \in \mathbb{R}^n$. Show that

$$\lim_{p\to\infty} \|\underline{x}\|_p = \|\underline{x}\|_\infty .$$

   (b) Show that the Euclidean vector norm and the Frobenius matrix norm are compatible, i.e.,

$$\|A\underline{x}\|_2 \le \|A\|_F \|\underline{x}\|_2 \quad \forall\, A \in \mathbb{R}^{m\times n}, \underline{x} \in \mathbb{R}^n.$$

4 points

3. *Finite Difference matrix for the second order derivative.* Consider the differential equation (Poisson equation, boundary value problem)

$$\begin{aligned} -u'' &= f \quad \text{in } (0,1), \\ u(0) &= a, \\ u(1) &= b. \end{aligned}$$

   Use an equidistant grid

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1, \quad h = x_i - x_{i-1},\ i = 1,\ldots,n,$$

   for the discretization of the second derivative.

(a) Show that the approximation (finite difference)

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} =: u_{xx,i}, \quad i = 1, \dots, n-1$$

is of second order consistent, i.e.,

$$u_{xx,i} = u''(x_i) + \mathcal{O}(h^2)$$

if $u \in C^4([0,1])$.

(b) Insert the approximation of the second order derivative in the differential equation and derive a linear system of equations for the values $u_i = u(x_i)$, $i = 0, \dots, n$. Set the boundary conditions and use as approximation for the right-hand side $f_i = f(x_i)$, $i = 1, \dots, n-1$.

(c) Reduce this linear system of equations to a system for u_i, $i = 1, \dots, n-1$, by eliminating the equations for the boundary conditions.

(d) Given n. Write a code that computes the matrix of the reduced system. If the programming language supports a `sparse` format, then store this matrix in this format.

This matrix is needed for further exercise problems! **4 points**

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, Oct. 21, 2024, 10:00 a.m.** via the whiteboard.