Appendix A
Topics on the Theory of Ordinary Differential Equations

A.1 Ordinary Differential Equations of Higher Order

Remark A.1. Motivation. The notation of stiffness comes from the consideration of first order systems of ordinary differential equations. There are some connections of such systems to ordinary differential equations of higher order, e.g. a solution method for linear first order systems requires the solution of a higher order linear differential equation, see Remark A.36. □

A.1.1 Definition, Connection to First Order Systems

Definition A.2. General and explicit n-th order ordinary differential equation. The general ordinary differential equation of order n has the form

$$F\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right) = 0. \tag{A.1}$$

This equation is called explicit, if one can write it in the form

$$y^{(n)}(x) = f\left(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right). \tag{A.2}$$

The function $y(x)$ is a solution of (A.1) in an interval I if $y(x)$ is n times continuously differentiable in I and if $y(x)$ satisfies (A.1).

Let $x_0 \in I$ be given. Then, (A.1) together with the conditions

$$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

is called initial value problem for (A.1). □

Example A.3. Special cases. The general resp. explicit ordinary differential equation of higher order can be solved analytically only in special cases. Two special cases, that will not be considered here, are as follows:

- Consider the second order differential equation

  $$y''(x) = f(x, y'(x)).$$

  Substituting $y'(x) = z(x)$, one obtains a first order differential equation for $z(x)$

  $$z'(x) = f(x, z(x)).$$

  If one can solve this equation analytically, one gets $y'(x)$. If it is then possible to find a primitive of $y'(x)$, one has computed an analytical solution of the differential equation of second order. In the case of an initial value problem with

  $$y(x_0) = y_0, \ y'(x_0) = y_1,$$

  the initial value for the first order differential equation is

  $$z(x_0) = y_1.$$

  The second initial value is needed for determining the constant of the primitive of $y'(x)$.
- Consider the differential equation of second order

  $$y''(x) = f(y, y').$$

  Let a solution $y(x)$ of this differential equation be known and let $y^{-1}(y)$ its inverse function, i.e. $y^{-1}(y(x)) = x$. Then, one can use the ansatz

  $$p(y) := y'\left(y^{-1}(y)\right).$$

  With the rule for differentiating the inverse function $((f^{-1})'(y_0) = 1/f'(x_0))$, one obtains

  $$\frac{dp}{dy}(y) = y''\left(y^{-1}(y)\right)\frac{d}{dy}\left(y^{-1}(y(x))\right) = \frac{y''\left(y^{-1}(y)\right)}{y'(x)} = \frac{y''\left(y^{-1}(y)\right)}{y'\left(y^{-1}(y)\right)}$$
  $$= \frac{y''\left(y^{-1}(y)\right)}{p(y)} = \frac{y''(x)}{p(y)}.$$

  This approach leads then to the first order differential equation

  $$p'(y) = \frac{f(y, p(y))}{p(y)}.$$

□

Theorem A.4. Connection of explicit ordinary differential equations of higher order and systems of differential equations of first order. *Every explicit differential equation of n-th order* (A.2) *can be transformed equivalently to a system of n differential equations of first order*

$$\begin{aligned} y_k'(x) &= y_{k+1}(x), \quad k = 1, \ldots, n-1, \\ y_n'(x) &= f(x, y_1(x), \ldots, y_n(x)) \end{aligned} \tag{A.3}$$

or (note that the system is generally nonlinear, since the unknown functions appear also in $f(\cdot, \ldots, \cdot)$)

$$\boldsymbol{y}'(x) = \begin{pmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(x, y_1, \ldots, y_n) \end{pmatrix}$$

for the n functions $y_1(x), \ldots, y_n(x)$. The solution of (A.2) *is $y(x) = y_1(x)$.*

Proof. Insert in (A.2)

$$\begin{aligned} &y_1(x) := y(x), \quad y_2(x) := y_1'(x) = y'(x), \quad y_3(x) := y_2'(x) = y''(x), \quad \ldots \\ &y_n(x) := y_{n-1}'(x) = y^{(n-1)}(x). \end{aligned}$$

If $y \in C^n(I)$ is a solution of (A.2), then $y_1(x), \ldots, y_n(x)$ is obviously a solution of (A.3) in I.

Conversely, if $y_1(x), \ldots, y_n(x) \in C^1(I)$ is a solution of (A.3), then it holds

$$\begin{aligned} &y_2(x) = y_1'(x), \quad y_3(x) = y_2'(x) = y_1''(x), \ldots, y_n(x) = y_1^{(n-1)}(x) \\ &y_n'(x) = y_1^{(n)}(x) = f(x, y_1, \ldots, y_n). \end{aligned}$$

Hence, the function $y_1(x)$ is n times continuously differentiable and it is the solution of (A.2) in I. ■

Example A.5. Transform of a higher order differential equation into a system of first order equations. The third order differential equation

$$y'''(x) + 2y''(x) - 5y'(x) = f(x, y(x))$$

can be transformed into the form

$$\begin{aligned} y_1(x) &= y(x) \\ y_1'(x) &= y_2(x)(= y'(x)) \\ y_2'(x) &= y_3(x)(= y''(x)) \\ y_3'(x) &= y'''(x) = -2y''(x) + 5y'(x) + f(x, y(x)) \\ &= -2y_3(x) + 5y_2(x) + f(x, y_1(x)). \end{aligned}$$

□

A.1.2 Linear Differential Equations of n-th Order

Definition A.6. Linear n-th order differential equations. A linear differential equation of n-th order has the form

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\ldots+a_1(x)y'(x)+a_0(x)y(x) = f(x), \quad \text{(A.4)}$$

where the functions $a_0(x), \ldots, a_n(x)$ are continuous in the interval I, in which a solution of (A.4) is searched, and it holds $a_n(x) \neq 0$ in I. The linear n-th order differential equation is called homogeneous if $f(x) = 0$ for all $x \in I$

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x) = 0. \quad \text{(A.5)}$$

□

Theorem A.7. Superposition principle for linear differential equations of higher order. *Consider the linear differential equation of n-th order* (A.4)*, then the superposition principle holds:*

i) If $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous equation (A.5)*, then $c_1y_1(x) + c_2y_2(x)$, $c_1, c_2 \in \mathbb{R}$, is a solution of the homogeneous equation, too.*
ii) If $y_0(x)$ is a solution of the inhomogeneous equation and $y_1(x)$ is a solution of the homogeneous equation, then $y_0(x) + y_1(x)$ is a solution of the inhomogeneous equation.
iii) If $y_1(x)$ and $y_2(x)$ are two solutions of the inhomogeneous equation, then $y_1(x) - y_2(x)$ is a solution of the homogeneous equation.

Proof. Direct calculations, exercise. ■

Corollary A.8. General solution of the inhomogeneous differential equation. *The general solution of* (A.4) *is the sum of the general solution of the homogeneous linear differential equation of n-th order* (A.5) *and one special solution of the inhomogeneous n-th order differential equation* (A.4)*.*

Remark A.9. Transform in a linear system of ordinary differential equations of first order. A linear differential equation of n-th order can be transformed equivalently into a linear $n \times n$ system

$$\begin{aligned} y_k'(x) &= y_{k+1}(x), \quad k = 1, \ldots, n-1, \\ y_n'(x) &= -\sum_{i=0}^{n-1} \frac{a_i(x)}{a_n(x)} y_{i+1}(x) + \frac{f(x)}{a_n(x)} \end{aligned}$$

or

$$\boldsymbol{y}'(x) = \begin{pmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{a_0(x)}{a_n(x)} & -\frac{a_1(x)}{a_n(x)} & -\frac{a_2(x)}{a_n(x)} & \cdots & -\frac{a_{n-1}(x)}{a_n(x)} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{f(x)}{a_n(x)} \end{pmatrix}$$

$$=: A(x)\boldsymbol{y}(x) + \boldsymbol{f}(x). \tag{A.6}$$

□

Theorem A.10. Existence and uniqueness of a solution of the initial value problem. *Let $I = [x_0 - a, x_0 + a]$ and $a_i \in C(I)$, $i = 0, \ldots, n$, $f \in C(I)$. Then, the linear differential equation of n-th order* (A.4) *has exactly one solution $y \in C^n(I)$ for given initial value*

$$y(x_0) = y_0, \; y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}.$$

Proof. Since (A.4) is equivalent to the system (A.6), one can apply the theorem on global existence and uniqueness of a solution of an initial value problem from Picard–Lindelöf, see lecture notes Numerical Mathematics I or the literature. To this end, one has to show the Lipschitz continuity of the right-hand side of (A.6) with respect to $y_1, \ldots, y_n$. Denoting the right-hand side by $F(x, \boldsymbol{y})$ gives

$$\|\mathbf{F}(x,\boldsymbol{y}) - \mathbf{F}(x,\tilde{\boldsymbol{y}})\|_{[C(I)]^n} = \|A(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|_{[C(I)]^n} \le \|A\|_{[C(I)]^n,\infty} \|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_{[C(I)]^n},$$

where one uses the triangle inequality to get

$$\begin{aligned} \|A_{i\cdot}\boldsymbol{y}\|_{C(I)} &= \max_{x\in I} \left| \sum_{j=1}^n a_{ij}(x) y_j(x) \right| \le \max_{x\in I} \sum_{j=1}^n |a_{ij}(x)| \max_{j=1,\ldots,n} \left\{ \max_{x\in I} |y_j(x)| \right\} \\ &= \|A_{i\cdot}\|_{C(I)} \|\boldsymbol{y}\|_{[C(I)]^n} \end{aligned}$$

for $i = 1, \ldots, n$. Now, one can choose

$$L = \|A\|_{[C(I)]^n,\infty} = \max_{x\in I} \left\{ \max \left\{ 1, \left| \frac{a_1(x)}{a_n(x)} \right| + \ldots + \left| \frac{a_{n-1}(x)}{a_n(x)} \right| \right\} \right\}.$$

All terms are bounded since I is closed (compact) and continuous functions are bounded on compact sets. ■

Definition A.11. Linearly independent solutions, fundamental system. The solutions $y_i(x) : I \to \mathbb{R}$, $i = 1, \ldots, k$, of (A.5) are called linearly independent if from

$$\sum_{i=1}^k c_i y_i(x) = 0, \quad \text{for all } x \in I, \; c_i \in \mathbb{R},$$

it follows that $c_i = 0$ for $i = 1, \ldots, k$. A set of n linearly independent solutions is called a fundamental system of (A.5). □

Definition A.12. Wronski[1] matrix, Wronski determinant. Let $y_i(x)$, $i = 1, \ldots, k$, be solutions of (A.5). The matrix

$$\mathcal{W}(x) = \begin{pmatrix} y_1(x) & \ldots & y_k(x) \\ y_1'(x) & \ldots & y_k'(x) \\ & \vdots & \\ y_1^{(n-1)}(x) & \ldots & y_k^{(n-1)}(x) \end{pmatrix}$$

is called Wronski matrix. For $k = n$ the Wronski determinant is given by $\det(\mathcal{W})(x) =: W(x)$. □

Lemma A.13. Properties of the Wronski matrix and Wronski determinant. *Let $I = [a, b]$ and let $y_1(x), \ldots, y_n(x)$ be solutions of* (A.5).

i) The Wronski determinant fulfills the linear first order differential equation

$$W'(x) = -\frac{a_{n-1}(x)}{a_n(x)} W(x).$$

ii) It holds for all $x \in I$

$$W(x) = W(x_0) \exp\left(-\int_{x_0}^{x} \frac{a_{n-1}(t)}{a_n(t)}\, dt\right)$$

with arbitrary $x_0 \in I$.

iii) If there exists a $x_0 \in I$ with $W(x_0) \neq 0$, then it holds $W(x) \neq 0$ for all $x \in I$.

iv) If there exists a $x_0 \in I$ with rank$(\mathcal{W}(x_0)) = k$, then there are at least k solutions of (A.5)*, e.g. $y_1(x), \ldots, y_k(x)$, linearly independent.*

Proof. i) Let S_n be the set of all permutations of $\{1, \ldots, n\}$ and let $\sigma \in S_n$. Denote the entries of the Wronski matrix by $\mathcal{W}(x) = \left(y_{jk}(x)\right)_{j,k=1}^{n}$. If $\sigma = (\sigma_1, \ldots, \sigma_n)$, then let

$$\prod_{j=1}^{n} y_{j,\sigma_j}(x) = \left(y_{1,\sigma_1} y_{2,\sigma_2} \ldots y_{n,\sigma_n}\right)(x).$$

Applying the Laplace[2] formula for determinants and the product rule yields

[1] Joseph Marie Wronski (1758 – 1853)

[2] Pierre–Simon (Marquis de) Laplace (1749 – 1827)

$$
\begin{aligned}
\frac{d}{dx}\det(\mathcal{W}(x)) &= \frac{d}{dx}\left(\sum_{\sigma\in S_n}\left(\operatorname{sgn}(\sigma)\prod_{j=1}^{n} y_{j,\sigma_j}(x)\right)\right)\\
&= \sum_{\sigma\in S_n}\left(\operatorname{sgn}(\sigma)\sum_{i=1}^{n}\left(\prod_{j=1,j\neq i}^{n} y_{j,\sigma_j}(x)\right)y'_{i,\sigma_i}(x)\right)\\
&= \sum_{i=1}^{n}\left(\sum_{\sigma\in S_n}\left(\operatorname{sgn}(\sigma)\prod_{j=1,j\neq i}^{n} y_{j,\sigma_j}(x)y'_{i,\sigma_i}(x)\right)\right)\\
&= \sum_{i=1}^{n}\det\begin{pmatrix} \ldots & \ldots & \ldots \\ \left(y_1^{(i-1)}(x)\right)' & \cdots & \left(y_n^{(i-1)}(x)\right)' \\ \ldots & \ldots & \ldots \end{pmatrix}.
\end{aligned}
$$

exercise for $n = 2, 3$. In the last step, again the Laplace formula for determinants was applied. In the i-th row of the last matrix is the first derivative of the corresponding row of the Wronski matrix, i.e. there is the i-th order derivative of $(y_1(x), \ldots, y_n(x))$. The rows with dots in this matrix coincide with the respective rows of $\mathcal{W}(x)$. For $i = 1, \ldots, n-1$, the determinants vanish, since in these cases there are two identical rows, namely row i and $i+1$. Thus, it is

$$
\frac{d}{dx}\det(\mathcal{W}(x)) = \det\begin{pmatrix} y_1(x) & \ldots & y_n(x) \\ y_1'(x) & \ldots & y_n'(x) \\ & \vdots & \\ y_1^{(n-2)}(x) & \ldots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & \ldots & y_n^{(n)}(x) \end{pmatrix}.
$$

Now, one uses that $y_1(x), \ldots, y_n(x)$ are solutions of (A.5) and one replaces the n-th derivative in the last row by (A.5). Using rules for the evaluation of determinants, one obtains

$$
\frac{d}{dx}\det(\mathcal{W}(x)) = \sum_{i=1}^{n} -\frac{a_{i-1}(x)}{a_n(x)}\det\begin{pmatrix} y_1(x) & \ldots & y_n(x) \\ y_1'(x) & \ldots & y_n'(x) \\ & \vdots & \\ y_1^{(i-1)}(x) & \ldots & y_n^{(i-1)}(x) \end{pmatrix}.
$$

Apart of the last term, all other determinants vanish, since all other terms have two identical rows, namely the i-th row and the last row.

ii) This term is the solution of the initial value problem for the Wronski determinant and the initial value $W(x_0)$, see the respective theorem in the lecture notes of Numerical Mathematics I.

iii) This statement follows directly from ii) since the exponential does not vanish.

iv) *exercise*

■

Theorem A.14. Existence of a fundamental system, representation of the solution of a homogeneous linear differential equation of n-th order by the fundamental system. *Let $I = [a, b]$ with $x_0 \in I$. The homogeneous equation* (A.5) *has a fundamental system in I. Each solution of* (A.5) *can be written as a linear combination of the solutions of an arbitrary fundamental system.*

Proof. Consider n homogeneous initial value problems with the initial values

$$y_j^{(i-1)}(x_0) = \delta_{ij}, \quad i,j = 1,\ldots,n.$$

Each of these initial value problems has a unique solution $y_j(x)$, see Theorem A.10. It is $W(x_0) = 1$ for these solutions. From Lemma A.13, iii), it follows that $\{y_1(x),\ldots,y_n(x)\}$ is a fundamental system.

Let $y(x)$ be an arbitrary solution of (A.5) with the initial values $y^{(i-1)}(x_0) = \tilde{y}_{i-1}$, $i = 1,\ldots,n$, and $\{y_1(x),\ldots,y_n(x)\}$ an arbitrary fundamental system. The system

$$\begin{pmatrix} y_1(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & \ldots & y_n'(x_0) \\ & \vdots & \\ y_1^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} \tilde{y}_0 \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{n-1} \end{pmatrix}$$

has a unique solution since the matrix spanned by a fundamental system is not singular. The function $\sum_{i=1}^{n} c_{i-1} y_i(x)$ satisfies the initial conditions (these are just the equations of the system) and, because of the superposition principle, it is a solution of (A.5). Since the solution of the initial value problem to (A.5) is unique, Theorem A.10, it follows that $y(x) = \sum_{i=1}^{n} c_{i-1} y_i(x)$. ■

Theorem A.15. Special solution of the inhomogeneous equation. *Let $\{y_1(x),\ldots,y_n(x)\}$ be a fundamental system of the homogeneous equation* (A.5) *in $I = [a,b]$. In addition, let $W_l(x)$ be the determinant, which is obtained from the Wronski determinant $W(x)$ with respect to $\{y_1(x),\ldots,y_n(x)\}$ by replacing the l-th column by $(0,0,\ldots,f(x)/a_n(x))^T$. Then,*

$$y(x) = \sum_{l=1}^{n} y_l(x) \int_{x_0}^{x} \frac{W_l(t)}{W(t)}\, dt, \quad x_0, x \in I,$$

is a solution of the inhomogeneous equation (A.4).

Proof. The proof uses the principle of the variation of the constants. This principle will be explained in a simpler setting in Remark A.27. For details of the proof, see the literature. ■

A.1.3 Linear n-th Order Differential Equations with Constant Coefficients

Definition A.16. Linear differential equation of n-th order with constant coefficients. A linear n-th order differential equation with constant coefficients has the form

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = f(x), \qquad \text{(A.7)}$$

with $a_i \in \mathbb{R}$, $i = 0,\ldots,n$, $a_n \neq 0$. □

A.1.3.1 The Homogeneous Equation

Remark A.17. Basic approach for solving the homogeneous linear differential equation of n-th order with constant coefficients. Because of the superposition principle, one needs the general solution of the homogeneous differential equation. That means, one has to find a fundamental system, i.e. n linearly independent solutions.

Consider

$$\sum_{i=0}^{n} a_i y_h^{(i)}(x) = 0. \qquad \text{(A.8)}$$

In the case of a differential equation of first order, i.e. $n = 1$,

$$a_1 y_h'(x) + a_0 y_h(x) = 0,$$

one can get the solution by the method of separating the variables (unknowns), see lecture notes of Numerical Mathematics I. One obtains

$$y_h(x) = c \exp\left(-\frac{a_0}{a_1}x\right), \quad c \in \mathbb{R}.$$

One uses the same structural ansatz for computing the solution of (A.8)

$$y_h(x) = e^{\lambda x}, \quad \lambda \in \mathbb{C}. \qquad \text{(A.9)}$$

It follows that

$$y_h'(x) = \lambda e^{\lambda x}, \ldots, y_h^{(n)}(x) = \lambda^n e^{\lambda x}.$$

Inserting into (A.8) gives

$$\left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0\right) e^{\lambda x} = 0. \qquad \text{(A.10)}$$

It is $e^{\lambda x} \neq 0$, also for complex λ. Because, using Euler's formula, it holds for $\lambda = a + ib$, $a, b \in \mathbb{R}$, that

$$e^{\lambda x} = e^{ax}\left(\cos(bx) + i\sin(bx)\right) = e^{ax}\cos(bx) + ie^{ax}\sin(bx).$$

A complex number is zero iff its real part and its imaginary part are vanish. It is $e^{ax} > 0$ and there does not exist a $(bx) \in \mathbb{R}$ such that at the same time $\sin(bx)$ and $\cos(bx)$ vanish. Hence, $e^{\lambda x} \neq 0$.

The equation (A.10) is satisfied iff one of the factors is equal to zero. Since the second factor cannot vanish, it must hold

$$p(\lambda) := a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0.$$

The function $p(\lambda)$ is called characteristic polynomial of (A.8). The roots of the characteristic polynomial are the values of λ in the ansatz of $y_h(x)$.

From the fundamental theorem of algebra it holds that $p(\lambda)$ has exactly n roots, which do not need to be mutually different. Since the coefficients of $p(\lambda)$ are real numbers, it follows that with each complex root $\lambda_1 = a + ib$, $a, b \in \mathbb{R}$, $b \neq 0$, also its conjugate $\lambda_2 = a - ib$ is a root of $p(\lambda)$.

It will be shown that the basic ansatz (A.9) is not sufficient in the case of multiple roots. □

Theorem A.18. Linearly independent solutions in the case of real roots with multiplicity k**.** *Let* $\lambda_0 \in \mathbb{R}$ *be a real root of the characteristic polynomial* $p(\lambda)$ *with multiplicity* k*,* $1 \le k \le n$*. Then, one can obtain with* λ_0 *the* k *linearly independent solutions of* (A.8)

$$y_{h,1}(x) = e^{\lambda_0 x},\ y_{h,2}(x) = x e^{\lambda_0 x},\ \ldots,\ y_{h,k}(x) = x^{k-1} e^{\lambda_0 x}. \tag{A.11}$$

Proof. For $k = 2$.

$y_{h,1}(x), y_{h,2}(x)$ *solve* (A.8). This statement is already clear for $y_{h,1}(x)$ since this function has the form of the ansatz (A.9). For $y_{h,2}(x)$ it holds

$$\begin{aligned} y'_{h,2}(x) &= (1 + \lambda_0 x)\, e^{\lambda_0 x}, \\ y''_{h,2}(x) &= \left(2\lambda_0 + \lambda_0^2 x\right) e^{\lambda_0 x}, \\ &\ \vdots \\ y^{(n)}_{h,2}(x) &= \left(n\lambda_0^{n-1} + \lambda_0^n x\right) e^{\lambda_0 x}. \end{aligned}$$

Inserting into the left-hand side of (A.8) yields

$$e^{\lambda_0 x} \sum_{i=0}^{n} a_i (i\lambda_0^{i-1} + \lambda_0^i x) = e^{\lambda_0 x} \left(x \underbrace{\sum_{i=0}^{n} a_i \lambda_0^i}_{p(\lambda_0)} + \underbrace{\sum_{i=0}^{n} a_i i \lambda_0^i}_{p'(\lambda_0)} \right). \tag{A.12}$$

It is $p(\lambda_0) = 0$, since λ_0 is a root of $p(\lambda)$. The second term is the derivative $p'(\lambda)$ of $p(\lambda)$ at λ_0. Since the multiplicity of λ_0 is two, one can write $p(\lambda)$ in the form

$$p(\lambda) = (\lambda - \lambda_0)^2\, p_0(\lambda),$$

where $p_0(\lambda)$ is a polynomial of degree $n - 2$. It follows that

$$p'(\lambda) = 2\,(\lambda - \lambda_0)\, p_0(\lambda) + (\lambda - \lambda_0)^2\, p_0'(\lambda).$$

Hence, it holds $p'(\lambda_0) = 0$, (A.12) vanishes, and $y_{h,2}(x)$ is a solution of (A.8).

$y_{h,1}(x), y_{h,2}(x)$ *are linearly independent.* One has to show, Lemma A.13, that the Wronski determinant does not vanish. It holds

$$\begin{aligned} W(x) &= \det \begin{pmatrix} y_{h,1}(x) & y_{h,2}(x) \\ y'_{h,1}(x) & y'_{h,2}(x) \end{pmatrix} = \det \begin{pmatrix} e^{\lambda_0 x} & x e^{\lambda_0 x} \\ \lambda_0 e^{\lambda_0 x} & (1 + \lambda_0 x) e^{\lambda_0 x} \end{pmatrix} \\ &= e^{2\lambda_0 x} \det \begin{pmatrix} 1 & x \\ \lambda_0 & 1 + \lambda_0 x \end{pmatrix} = e^{2\lambda_0 x}\,(1 + \lambda_0 x - \lambda_0 x) = e^{2\lambda_0 x} > 0 \end{aligned}$$

for all $x \in I$.

Roots of multiplicity $k > 2$. The principle proof is analogous to the case $k = 2$, where one uses the factorization $p(\lambda) = (\lambda - \lambda_0)^k p_0(\lambda)$. The computation of the Wronski determinant becomes more involved. ∎

Remark A.19. Complex roots. The statement of Theorem A.18 is true also for complex roots of $p(\lambda)$. The Wronski determinant is $e^{2\lambda_1 x} \neq 0$. However, the corresponding solutions, e.g.

$$\tilde{y}_{1,h}(x) = e^{\lambda_1 x} = e^{(a+ib)x}$$

are complex-valued. Since one has real coefficients in (A.8), one likes to obtain also real-valued solutions. Such solutions can be constructed from the complex-valued solutions.

Let $\lambda_1 = a + ib$, $\overline{\lambda_1} = a - ib$, $a, b \in \mathbb{R}$, $b \neq 0$, be a conjugate complex roots of $p(\lambda)$, then one obtains with Euler's formula

$$\begin{aligned} e^{\lambda_1 x} &= e^{(a+ib)x} = e^{ax} \left(\cos(bx) + i \sin(bx)\right), \\ e^{\overline{\lambda_1} x} &= e^{(a-ib)x} = e^{ax} \left(\cos(bx) - i \sin(bx)\right). \end{aligned}$$

Because of the superposition principle, each linear combination is also solution of (A.8). □

Theorem A.20. Linearly independent solution for simple conjugate complex roots. *Let $\lambda_1 \in \mathbb{C}$, $\lambda_1 = a+ib$, $b \neq 0$, be a simple conjugate complex root of the characteristic polynomial $p(\lambda)$ with real coefficients. Then,*

$$y_{h,1}(x) = Re\left(e^{\lambda_1 x}\right) = e^{ax}\cos(bx), \quad y_{h,2}(x) = Im\left(e^{\lambda_1 x}\right) = e^{ax}\sin(bx),$$

are real-valued, linearly independent solutions of (A.8).

Proof. Use the superposition principle for proving that the functions are solutions and the Wronski determinant for proving that they are linearly independent, exercise. ∎

Theorem A.21. Linearly independent solution for conjugate complex roots with multiplicity greater than one. *Let $\lambda_1 \in \mathbb{C}$, $\lambda_1 = a + ib$, $b \neq 0$, be a conjugate complex root with multiplicity k of the characteristic polynomial $p(\lambda)$ with real coefficients. Then,*

$$\begin{aligned} y_{h,1}(x) = e^{ax}\cos(bx), \ldots, y_{h,k}(x) = x^{k-1}e^{ax}\cos(bx), \\ y_{h,k+1}(x) = e^{ax}\sin(bx), \ldots, y_{h,2k}(x) = x^{k-1}e^{ax}\sin(bx) \end{aligned} \tag{A.13}$$

are real-valued, linearly independent solutions of (A.8).

Proof. The proof is similarly to the previous theorems. ∎

Theorem A.22. Fundamental system for (A.8)**.** *Let $p(\lambda)$ be the characteristic polynomial of* (A.8) *with the roots $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, where the roots are counted in correspondence to their multiplicity. Then, the set of solutions of form* (A.11) *and* (A.13) *form a fundamental system of* (A.8).

Proof. A real root with multiplicity k gives k linearly independent solutions and a conjugate complex root with multiplicity k gives $2k$ linearly independent solutions. Thus, the total number of solutions of form (A.11) and (A.13) is equal to the number of roots of $p(\lambda)$. This number is equal to n, because of the fundamental theorem of algebra. It is known from Theorem A.14 that a fundamental system has exactly n functions. Altogether, the correct number of functions is there.

One can show that solutions that correspond to different roots are linearly independent, e.g., (Günther *et al.*, 1974, p. 75). The linearly independence of the solutions that belong to the same root, was already proved. ■

Example A.23. Homogeneous second order linear differential equation with constant coefficients.

1. Consider
$$y''(x) + 6y'(x) + 9y(x) = 0.$$
The characteristic polynomial is
$$p(\lambda) = \lambda^2 + 6\lambda + 9$$
with the roots $\lambda_1 = \lambda_2 = -3$. One obtains the fundamental system
$$y_{h,1}(x) = e^{-3x}, \quad y_{h,2}(x) = xe^{-3x}.$$
The general solution of the homogeneous equation has the form
$$y_h(x) = c_1 y_{h,1}(x) + c_2 y_{h,2}(x) = c_1 e^{-3x} + c_2 x e^{-3x}, \quad c_1, c_2 \in \mathbb{R}.$$
2. Consider
$$y''(x) + 4y(x) = 0 \quad \Longrightarrow \quad p(\lambda) = \lambda^2 + 4 \quad \Longrightarrow \quad \lambda_{1,2} = \pm 2i.$$
It follows that
$$\begin{aligned} y_{h,1}(x) &= \cos(2x), \quad y_{h,2}(x) = \sin(2x) \\ y_h(x) &= c_1 \cos(2x) + c_2 \sin(2x), \quad c_1, c_2 \in \mathbb{R}. \end{aligned}$$

□

A.1.3.2 The Inhomogeneous Equation

Remark A.24. Goal. Because of the superposition principle, a special solution of (A.7) has to be found. This section sketches several possibilities to obtain such a solution. □

Remark A.25. Appropriate ansatz (Störgliedansätze). If the right-hand side $f(x)$ possesses a special form, it is possible to obtain a solution of the inhomogeneous equation (A.7) with an appropriate ansatz. From (A.7) it becomes clear, that this way works only if on the left-hand side and the right-hand side of the equation are the same types of functions. In particular, one needs the same types of functions for $y_i(x)$ and all derivatives up to order n. This approach works, e.g., for the following classes of right-hand sides:

- $f(x)$ is a polynomial

$$f(x) = b_0 + b_1 x + \ldots + b_m x^m, \quad b_m \neq 0.$$

  The appropriate ansatz is also a polynomial

$$y_i(x) = x^k \left(c_0 + c_1 x + \ldots + c_m x^m \right),$$

  where 0 is a root of $p(\lambda)$ with multiplicity k.
- If the right-hand side is

$$f(x) = \left(b_0 + b_1 x + \ldots + b_m x^m \right) e^{ax},$$

  then one can use the following ansatz

$$y_i(x) = x^k \left(c_0 + c_1 x + \ldots + c_m x^m \right) e^{ax},$$

  where a is a root of $p(\lambda)$ with multiplicity k. The first class of functions is just a special case for $a = 0$.
- For right-hand sides of the form

$$\begin{aligned} f(x) &= \left(b_0 + b_1 x + \ldots + b_m x^m \right) \cos(bx), \\ f(x) &= \left(b_0 + b_1 x + \ldots + b_m x^m \right) \sin(bx), \end{aligned}$$

  one can use the ansatz

$$\begin{aligned} y_i(x) = {} & x^k \left(c_0 + c_1 x + \ldots + c_m x^m \right) \cos(bx) \\ & + x^k \left(d_0 + d_1 x + \ldots + d_m x^m \right) \sin(bx), \end{aligned}$$

  if ib is a root of $p(\lambda)$ with multiplicity k.

One can find the ansatz for more right-hand sides in the literature, e.g. in Heuser (2006). □

Example A.26. Appropriate ansatz (Störgliedansatz). Consider

$$y''(x) - y'(x) + 2y(x) = \cos x.$$

The appropriate ansatz is given by

$$\begin{aligned} y_i(x) &= a \cos x + b \sin x \quad \Longrightarrow \\ y_i'(x) &= -a \sin x + b \cos x \quad \Longrightarrow \\ y_i''(x) &= -a \cos x - b \sin x. \end{aligned}$$

Inserting into the equation gives

$$-a\cos x - b\sin x + a\sin x - b\cos x + 2a\cos x + 2b\sin x = \cos x \implies$$
$$(-a - b + 2a)\cos x + (-b + a + 2b)\sin x = \cos x.$$

The last equation is satisfied if the numbers a, b solve the following linear system of equations

$$a - b = 1,\ a + b = 0 \quad \implies \quad a = \frac{1}{2},\ b = -\frac{1}{2}.$$

One obtains the special solution

$$y_i(x) = \frac{1}{2}(\cos x - \sin x).$$

□

Remark A.27. Variation of the constants. If one cannot find an appropriate ansatz, then one can try the variation of the constants. This approach will be demonstrated for the second order differential equation

$$y''(x) + a_1 y'(x) + a_0 y(x) = f(x). \tag{A.14}$$

Let $y_{h,1}(x), y_{h,2}(x)$ be two linearly independent solutions of the homogeneous differential equation such that

$$y_h(x) = c_1 y_{h,1}(x) + c_2 y_{h,2}(x)$$

is the general solution of the homogeneous equation. Now, one makes the ansatz

$$y_i(x) = c_1(x) y_{h,1}(x) + c_2(x) y_{h,2}(x)$$

with two unknown functions $c_1(x), c_2(x)$. The determination of these functions requires two conditions. One has

$$\begin{aligned} y_i'(x) &= c_1'(x) y_{h,1}(x) + c_1(x) y_{h,1}'(x) + c_2'(x) y_{h,2}(x) + c_2(x) y_{h,2}'(x) \\ &= (c_1'(x) y_{h,1}(x) + c_2'(x) y_{h,2}(x)) + c_1(x) y_{h,1}'(x) + c_2(x) y_{h,2}'(x). \end{aligned}$$

Now, one sets the term in the parentheses zero. This is the first condition. It follows that

$$y_i''(x) = c_1'(x) y_{h,1}'(x) + c_1(x) y_{h,1}''(x) + c_2'(x) y_{h,2}'(x) + c_2(x) y_{h,2}''(x).$$

Inserting this expression into (A.14) gives

$$
\begin{aligned}
f(x) &= c_1'(x)y_{h,1}'(x) + c_1(x)y_{h,1}''(x) + c_2'(x)y_{h,2}'(x) + c_2(x)y_{h,2}''(x) \\
&\quad + a_1\left(c_1(x)y_{h,1}'(x) + c_2(x)y_{h,2}'(x)\right) + a_0\left(c_1(x)y_{h,1}(x) + c_2(x)y_{h,2}(x)\right) \\
&= c_1(x)\underbrace{\left(y_{h,1}''(x) + a_1 y_{h,1}'(x) + a_0 y_{h,1}(x)\right)}_{=0} \\
&\quad + c_2(x)\underbrace{\left(y_{h,2}''(x) + a_1 y_{h,2}'(x) + a_0 y_{h,2}(x)\right)}_{=0} \\
&\quad + c_1'(x)y_{h,1}'(x) + c_2'(x)y_{h,2}'(x).
\end{aligned}
$$

This is the second condition. Summarizing both conditions gives the following system of equations

$$
\begin{pmatrix} y_{h,1}(x) & y_{h,2}(x) \\ y_{h,1}'(x) & y_{h,2}'(x) \end{pmatrix} \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}.
$$

This system possesses a unique solution since $y_{h,1}(x), y_{h,2}(x)$ are linearly independent from what follows that the determinant of the system matrix, which is just the Wronksi matrix, is not equal to zero. The solution is

$$
c_1'(x) = -\frac{f(x)y_{h,2}(x)}{y_{h,1}(x)y_{h,2}'(x) - y_{h,1}'(x)y_{h,2}(x)}, \; c_2'(x) = \frac{f(x)y_{h,1}(x)}{y_{h,1}(x)y_{h,2}'(x) - y_{h,1}'(x)y_{h,2}(x)}.
$$

The success of the method of the variation of the constants depends only on the difficulty to find the primitives of $c_1'(x)$ and $c_2'(x)$.

For equations of order higher than two, one has the goal to get a linear system of equations for $c_1'(x), \ldots, c_n'(x)$. To this end, one sets for each derivative of the ansatz the terms with $c_1'(x), \ldots, c_n'(x)$ equal to zero. The obtained linear system of equations has as matrix the Wronski matrix and as right-hand side a vector, whose first $(n-1)$ components are equal to zero and whose last component is $f(x)$. □

Example A.28. Variation of the constants. Find the general solution of

$$
y''(x) + 6y'(x) + 9y(x) = \frac{e^{-3x}}{1+x}.
$$

The general solution of the homogeneous equation is

$$
y_h(x) = c_1 e^{-3x} + c_2 x e^{-3x},
$$

see Example A.23. The variation of the constants leads to the following system of linear equations

$$
\begin{pmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & (1-3x)e^{-3x} \end{pmatrix} \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^{-3x}}{1+x} \end{pmatrix}.
$$

Using, e.g., the Cramer rule, gives

$$
\begin{aligned}
c_1'(x) &= -\frac{e^{-6x}\left(\frac{x}{1+x}\right)}{(1-3x+3x)e^{-6x}} = -\frac{x}{1+x}, \\
c_2'(x) &= \frac{e^{-6x}\left(\frac{1}{1+x}\right)}{(1-3x+3x)e^{-6x}} = \frac{1}{1+x}.
\end{aligned}
$$

One obtains

$$
\begin{aligned}
c_1(x) &= -\int \frac{x}{1+x}\,dx = -\int \frac{1+x}{1+x}\,dx + \int \frac{1}{1+x}\,dx = -x + \ln|1+x|\,, \\
c_2(x) &= \int \frac{1}{1+x}\,dx = \ln|1+x|\,.
\end{aligned}
$$

Thus, one gets

$$
y_i(x) = (-x + \ln|1+x|)\, e^{-3x} + \ln|1+x|\, x e^{-3x}
$$

and one obtains for the general solution

$$
y(x) = (-x + \ln|1+x| + c_1)\, e^{-3x} + (\ln|1+x| + c_2)\, x e^{-3x}.
$$

Inserting this function into the equation proves the correctness of the result.
□

A.2 Linear Systems of Ordinary Differential Equations of First Order

A.2.1 Definition, Existence and Uniqueness of a Solution

Definition A.29. Linear system of first order differential equations. In a linear system of ordinary differential equations of first order one tries to find functions $y_1(x), \ldots, y_n(x) \; : \; I \to \mathbb{R}$, $I = [a,b] \subset \mathbb{R}$, that satisfy the system

$$
y_i'(x) = \sum_{j=1}^{n} a_{ij}(x) y_j(x) + f_i(x), i = 1, \ldots, n,
$$

or in matrix-vector notation

$$
\boldsymbol{y}'(x) = A(x)\boldsymbol{y}(x) + \boldsymbol{f}(x) \tag{A.15}
$$

with

$$\boldsymbol{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \ \boldsymbol{y}'(x) = \begin{pmatrix} y_1'(x) \\ \vdots \\ y_n'(x) \end{pmatrix},$$

$$A(x) = \begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{pmatrix}, \ \boldsymbol{f}(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix},$$

where $a_{ij}(x), f_i(x) \in C(I)$. If $\boldsymbol{f}(x) \equiv \boldsymbol{0}$, then the system is called homogeneous. □

Theorem A.30. Superposition principle for linear systems. *Consider the linear system of ordinary differential equations* (A.15)*, then the superposition principle holds:*

i) If $\boldsymbol{y}_1(x)$ and $\boldsymbol{y}_2(x)$ are two solutions of the homogeneous systems, then $c_1\boldsymbol{y}_1(x) + c_2\boldsymbol{y}_2(x)$, $c_1, c_2 \in \mathbb{R}$, is a solution of the homogeneous system, too.

ii) If $\boldsymbol{y}_0(x)$ is a solution of the inhomogeneous system and $\boldsymbol{y}_1(x)$ is a solution of the homogeneous system, then $\boldsymbol{y}_0(x) + \boldsymbol{y}_1(x)$ is a solution of the inhomogeneous system.

iii) If $\boldsymbol{y}_1(x)$ and $\boldsymbol{y}_2(x)$ are two solutions of the inhomogeneous system, then $\boldsymbol{y}_1(x) - \boldsymbol{y}_2(x)$ is a solution of the homogeneous system.

Proof. Direct calculations, exercise. ■

Corollary A.31. General solution of the inhomogeneous system.

i) If $\boldsymbol{y}_1(x), \boldsymbol{y}_2(x), \ldots, \boldsymbol{y}_k(x)$ are solutions of the homogeneous system, then any linear combination $\sum_{i=1}^{k} c_i \boldsymbol{y}_i(x)$, $c_1, \ldots, c_k \in \mathbb{R}$, is also a solution of the homogeneous system.

ii) The general solution of the inhomogeneous system is the sum of a special solution of the inhomogeneous system and the general solution of the homogeneous system.

Theorem A.32. Existence and uniqueness of a solution of the initial value problem. *Let $I = [x_0 - a, x_0 + a]$ and $a_{ij} \in C(I)$, $f_i \in C(I)$, $i, j = 1, \ldots, n$. Then, there is exactly one solution $\boldsymbol{y}(x) : I \to \mathbb{R}^n$ of the initial value problem to* (A.15) *with the initial value $\boldsymbol{y}(x_0) = \boldsymbol{y}_0 \in \mathbb{R}^n$.*

Proof. The statement of the theorem follows from the theorem on global existence and uniqueness of a solution of an initial value problem from Picard–Lindelöf, see lecture notes Numerical Mathematics I or the literature.

Since the functions $a_{ij}(x)$ are continuous on the closed (compact) interval I, they are also bounded due to the Weierstrass theorem. That means, there is a constant M with

$$|a_{ij}(x)| \le M, \quad x \in I, \ i, j = 1, \ldots, n.$$

Denoting the right hand side of (A.15) by $\boldsymbol{f}(x, \boldsymbol{y})$, it follows that

$$\begin{aligned}
\|\boldsymbol{f}(x,\boldsymbol{y}_1)-\boldsymbol{f}(x,\boldsymbol{y}_2)\|_\infty &= \max_{i=1,\ldots,n} |f_i(x,\boldsymbol{y}_1)-f_i(x,\boldsymbol{y}_2)| \\
&= \max_{i=1,\ldots,n} \left| \sum_{j=1}^n a_{ij}(x)y_{1,j}(x) + f_i(x) - \sum_{j=1}^n a_{ij}(x)y_{2,j}(x) - f_i(x) \right| \\
&= \max_{i=1,\ldots,n} \left| \sum_{j=1}^n a_{ij}(x)\left(y_{1,j}(x)-y_{2,j}(x)\right) \right| \\
&\le n \max_{i,j=1,\ldots,n} |a_{ij}(x)| \max_{i=1,\ldots,n} |y_{1,i}(x)-y_{2,i}(x)| \\
&\le nM \|\boldsymbol{y}_1-\boldsymbol{y}_2\|_\infty ,
\end{aligned}$$

i.e. the right hand side satisfies a uniform Lipschitz condition with respect to $\boldsymbol{y}$ with the Lipschitz constant nM. Hence, the assumptions of the theorem on global existence and uniqueness of a solution of an initial value problem from Picard–Lindelöf are satisfied. ■

A.2.2 Solution of the Homogeneous System

Remark A.33. Scalar case. Because of the superposition principle, one needs the general solution of the homogeneous system

$$\boldsymbol{y}'(x) = A(x)\boldsymbol{y}(x) \tag{A.16}$$

for finding the general solution of (A.15). The homogeneous system has always the trivial solution $\boldsymbol{y}(x) = \boldsymbol{0}$.

In the scalar case $y'(x) = a(x)y(x)$, the general solution has the form

$$y(x) = c\exp\left(\int_{x_0}^x a(t)\ dt\right), \quad c \in \mathbb{R}, x_0 \in (a,b),$$

see lecture notes Numerical Mathematics I or the literature. Also for the system (A.16), it is possible to specify the general solution with the help of the exponential. □

Theorem A.34. General solution of the homogeneous linear system of first order. *The general solution of* (A.16) *is*

$$\boldsymbol{y}_h(x) = e^{\int_{x_0}^x A(t)\ dt}\boldsymbol{c}, \quad \boldsymbol{c} \in \mathbb{R}^n, x_0 \in (a,b). \tag{A.17}$$

The integral is defined component-wise.

Proof. *i)* (A.17) *is a solution of* (A.16). This statement follows from the derivative of the matrix exponential and the rule on the differentiation of an integral with respect to the upper limit

$$\boldsymbol{y}_h'(x) = \frac{d}{dx}\left(e^{\int_{x_0}^x A(t)\ dt}\boldsymbol{c}\right) = \frac{d}{dx}\left(\int_{x_0}^x A(t)\ dt\right)e^{\int_{x_0}^x A(t)\ dt}\boldsymbol{c} = A(x)e^{\int_{x_0}^x A(t)\ dt}\boldsymbol{c}.$$

ii) every solution of (A.16) *is of form* (A.17). Consider an arbitrary solution $\tilde{\boldsymbol{y}}_h(x)$ of (A.16) with $\tilde{\boldsymbol{y}}_h(x_0) \in \mathbb{R}^n$. Take in (A.17) $\boldsymbol{c} = \tilde{\boldsymbol{y}}_h(x_0)$. Then, it follows that

$$\boldsymbol{y}_h(x_0) = e^{\int_{x_0}^{x_0} A(t)\ dt} \tilde{\boldsymbol{y}}_h(x_0) = \underbrace{e^0}_{=I} \tilde{\boldsymbol{y}}_h(x_0) = \tilde{\boldsymbol{y}}_h(x_0).$$

That means, $e^{\int_{x_0}^{x} A(t)\ dt} \tilde{\boldsymbol{y}}_h(x_0)$ is a solution of (A.16) which has in x_0 the same initial value as $\tilde{\boldsymbol{y}}_h(x)$. Since the solution of the initial value problem is unique, Theorem A.32, it follows that $\tilde{\boldsymbol{y}}_h(x) = e^{\int_{x_0}^{x} A(t)\ dt} \tilde{\boldsymbol{y}}_h(x_0)$. ■

A.2.3 Linear Systems of First Order with Constant Coefficients

Remark A.35. Linear system of first order differential equations with constant coefficients. A linear system of first order differential equations with constant coefficients has the form

$$\boldsymbol{y}'(x) = A\boldsymbol{y}(x) + \boldsymbol{f}(x), \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{A.18}$$

Thus, the homogeneous system has the form

$$\boldsymbol{y}'(x) = A\boldsymbol{y}(x). \tag{A.19}$$

Its general solution is given by

$$\boldsymbol{y}_h(x) = e^{Ax}\boldsymbol{c}, \quad \boldsymbol{c} \in \mathbb{R}^n, \tag{A.20}$$

see Theorem A.34. □

Remark A.36. Elimination method, substitution method for the homogeneous system. One needs, due to the superposition principle, the general solution of the homogeneous system. In practice, it is generally hard to compute $\exp(Ax)$ because it is defined by an infinity series. For small systems, i.e. $n \le 3, 4$, one can use the elimination or substitution method for computing the general solution of (A.19). This method is already known from the numerical solution of linear systems of equations. One solves one equation for a certain unknown function $y_i(x)$ and inserts the result into the other equations. For differential equations, the equation has to be differentiated, see Example A.37. This step reduces the dimension of the system by one. One continues with this method until one reaches an equation with only one unknown function. For this function, a homogeneous linear differential equation of order n has to be solved, see Section A.1.3. The other components of the solution vector of (A.19) can be obtained by back substitution. □

Example A.37. Elimination method, substitution method. Find the solution of

$$\boldsymbol{y}'(x) = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \boldsymbol{y}(x) \iff y_1'(x) = -3y_1(x) - y_2(x),\ y_2'(x) = y_1(x) - y_2(x).$$

Solving the second equation for $y_1(x)$ and differentiating gives

$$y_1(x) = y_2'(x) + y_2(x), \quad y_1'(x) = y_2''(x) + y_2'(x).$$

Inserting into the first equation yields

$$y_2''(x) + y_2'(x) = -3\left(y_2'(x) + y_2(x)\right) - y_2(x) \iff y_2''(x) + 4y_2'(x) + 4y_2(x) = 0.$$

The general solution of this equation is

$$y_2(x) = c_1 e^{-2x} + c_2 x e^{-2x}, \quad c_1, c_2 \in \mathbb{R}.$$

One obtains from the second equation

$$y_1(x) = y_2'(x) + y_2(x) = \left(-c_1 + c_2\right) e^{-2x} - c_2 x e^{-2x}.$$

Thus, the general solution of the given linear system of differential equations with constant coefficients is computed by

$$\boldsymbol{y} = \begin{pmatrix} -c_1 + c_2 \\ c_1 \end{pmatrix} e^{-2x} + \begin{pmatrix} -c_2 \\ c_2 \end{pmatrix} x e^{-2x}.$$

Note that one can choose the constants in $y_2(x)$, but the constants in $y_1(x)$ are determined by the back substitution. If the constants should be choosen by $y_1(x)$, one obtains

$$\boldsymbol{y} = \begin{pmatrix} C_1 \\ C_2 - C_1 \end{pmatrix} e^{-2x} + \begin{pmatrix} C_2 \\ -C_2 \end{pmatrix} x e^{-2x}.$$

If an initial condition is given, then corresponding constants can be determined. □

Remark A.38. Other methods for computing the general solution of the homogeneous system. There are also other methods for computing the general solution of (A.19).

- The idea of the method of main-vectors and eigenvectors consists in transforming the system to a triangular system. Then it is possible to solve the equations successively. To this end, one constructs with the so-called main-vectors and eigenvectors an invertible matrix $C \in \mathbb{R}^{n\times n}$ such that $C^{-1}AC$ is a triangular matrix. One can show that such a matrix C exists for each $A \in \mathbb{R}^{n\times n}$. Then, one sets

$$\boldsymbol{y}(x) = C\boldsymbol{z}(x) \quad \Longrightarrow \quad \boldsymbol{y}'(x) = C\boldsymbol{z}'(x).$$

Inserting into (A.19) yields

$$C\boldsymbol{z}'(x) = AC\boldsymbol{z}(x) \quad \Longleftrightarrow \quad \boldsymbol{z}'(x) = C^{-1}AC\boldsymbol{z}(x).$$

This is a triangular system for $\boldsymbol{z}(x)$, which is solved successively for the components of $\boldsymbol{z}(x)$. The solution of (A.19) is obtained by computing $C\boldsymbol{z}(x)$.

- The method of matrix functions is based on an appropriate ansatz for the solution.

However, the application of both methods becomes very time-consuming for larger n, see the literature. □

Remark A.39. Methods for determining a special solution of the inhomogeneous system. For computing the general solution of the inhomogeneous system of linear differential equations of first order with constant coefficients, one needs also a special solution of the inhomogeneous system. There are several possibilities for obtaining this solution:

- *Method of the variation of constants.* One replaces $\boldsymbol{c}$ in (A.20) by $\boldsymbol{c}(x)$, inserts this expression into (A.18), obtains conditions for $\boldsymbol{c}'(x)$, and tries to compute $\boldsymbol{c}(x)$ from these conditions.
- *Appropriate ansatz (Störgliedansätze).* If each component of the right hand side $\boldsymbol{f}(x)$ has a special form, e.g., a polynomial, sine, cosine, or exponential, then it is often possible to find the special solution with an appropriate ansatz.
- *Method of elimination.* If the right hand side of $\boldsymbol{f}(x)$ of (A.18) is $(n-1)$ times continuously differentiable, then one can proceed exactly as in the elimination method. One obtains for one component of $\boldsymbol{y}(x)$ an inhomogeneous ordinary differential equation of order n with constant coefficients, for which one has to find a special solution. A special solution for (A.18) is obtained by back substitution.

□