



The MINI Element for the Stokes Equation

Master's Thesis submitted for the degree of Master of Science
(M.Sc.) at the Department of Mathematics and Computer
Science of Freie Universität Berlin

Gauri Singh Kshetry

Supervisor: Prof. Dr. Volker John
Second Reviewer: PD Dr. Alfonso Caiazzo

September 25, 2026

Declaration of Authorship

I, Gauri Singh Kshetry, hereby declare that I have written this thesis independently and that I have used no sources or aids other than those cited. All works that have been taken from other works, either directly or indirectly, have been clearly marked and properly referenced.

During the preparation of this thesis, AI-assisted tools, such as ChatGPT and Claude AI were solely to improve the readability, language quality, and formatting of the text. After using these tools, I carefully reviewed and edited the content. I take full responsibility for all mathematical content, analysis, arguments, and conclusions presented in this thesis.

Gauri Singh Kshetry

25.09.2026

Abstract

The Stokes equations provide a fundamental mathematical model for incompressible viscous flow, arising as a simplified form of the Navier–Stokes equations in the steady, viscosity dominated regime. Their finite element approximation leads to a mixed saddle point problem whose stability is governed by the Ladyzhenskaya–Babuška–Brezzi (inf-sup) condition. This thesis studies the MINI element, a low order, computationally inexpensive pair of finite element spaces that enriches the continuous piecewise linear velocity space with local bubble functions while retaining a continuous piecewise linear pressure space.

The thesis begins with the weak formulation of the Stokes equations, the continuous inf-sup condition, and existence, uniqueness, and stability of the weak solution. It then presents the construction of the MINI element and, via a Fortin operator built from a bubble corrected Clément interpolant, proves that the MINI element satisfies the discrete inf-sup condition independently of the mesh size. Finite element error estimates for the velocity in the H^1 -seminorm and for the pressure in the L^2 -norm are then derived, with particular attention to the pressure approximation and its dependence on the viscosity parameter. A Schur complement interpretation, following Pacheco and Steinbach, explains why pressure convergence rates above the classical first order estimate can appear on relatively coarse meshes.

These theoretical results are examined through manufactured solution experiments in two and three dimensions, most of the viscosity coefficients at $\nu = 1$ and $\nu = 10^{-6}$. Two-dimensional tests on a smooth solution and one satisfying the refined pressure analysis both show enhanced pressure convergence, approaching rates of ≈ 1.5 and ≈ 2 , respectively, as viscosity decreases, though lower viscosity also sharply increases velocity error magnitude. A third test, using a pressure field with a sharp localized transition and large H^2 -seminorm, delays this enhancement: convergence stays near first order until the finest meshes, largely regardless of viscosity, suggesting pressure regularity, not viscosity, governs when enhancement appears. A three-dimensional solution confirms the same picture, with near first order H^1 and second order L^2 velocity convergence, along with enhanced pressure rates of $\approx 1.7 - 1.85$.

Overall, the MINI element shows the expected velocity convergence, while its pressure approximation can exhibit enhanced pre-asymptotic convergence that depends on the pressure field’s regularity and structure rather than on viscosity, which instead mainly governs the size of the velocity error. These findings highlight the gap between the element’s classical asymptotic theory and the higher order convergence behavior observed in practice.

Acknowledgements

I would like to express my deepest and most sincere gratitude to my supervisor, Univ.-Prof. **Dr. Volker John**, for his guidance, patience, support, and encouragement throughout the preparation of this thesis. His expertise in numerical analysis, finite element methods, and incompressible flow problems has profoundly influenced my understanding of the subject and the development of this work. His book, *Finite Element Methods for Incompressible Flow Problems*, was an invaluable reference throughout this thesis and provided much of the theoretical foundation for my work. I am especially grateful for his guidance throughout the writing process, his thoughtful feedback, and his willingness to point me in the right direction whenever I needed it. I feel very fortunate to have had the opportunity to work under his supervision and to learn from his knowledge and experience. I would also like to express my sincere thanks to **Dr. Alfonso Caiazzo** for serving as the second reviewer of this thesis and for his time and consideration.

I am particularly grateful to **Dr. Dirk Peschka (WIAS)** for his generous help with the three-dimensional numerical experiment which made it possible to include this part of the numerical investigation within the scope of this thesis. I sincerely appreciate his time, effort, and support.

I have been fortunate to have my brother, **Gaurav Singh Kshetry**, by my side. As a physics student himself, he has been able to understand and appreciate many of the struggles I have faced. Despite having his own challenges and ambitions, he has always encouraged me to do my best, to take on every challenge with confidence, and to strive for excellence. Now, as he pursues his own path towards a **PhD in Physics**, I am incredibly proud of him. He reminded me of my own capabilities whenever I forgot them. I could not have asked for a better brother, and I am deeply grateful to have had him beside me throughout this journey.

My deepest gratitude goes to my parents, **Jeetendra Singh Kshetry** and **Sonu Singh Kshetry**, who have been my backbone throughout my life. Although they are back in India, their love, encouragement, and unwavering support have always reached me. My brother and my parents are the pillars of my foundation. One of the things I will always cherish most is how my brother and my parents have celebrated every achievement of mine, no matter how small it may have seemed to anyone else. For them, every small success was worth celebrating. Their happiness in my achievements has often been greater than my own. Whatever success I have achieved so far and whatever I may achieve in the future is in large part because of them and for them. I hope that I can continue to make them proud and achieve not only what they have dreamed for me, but also much more than they could have imagined.

Contents

Contents	i
1 Introduction	1
2 The Stokes Equations	3
2.1 Linearity	4
2.2 The Weak Formulation	4
2.3 Inf-Sup Condition	6
2.4 Existence and Uniqueness of the Weak Solution	7
2.5 Stability of the Solution of the Stokes Equations	7
2.6 Inf-sup (LBB) Condition	9
3 Discretization with the MINI Element	14
3.1 MINI Element	14
3.2 Fortin Operator	17
3.3 Discrete Inf-Sup Condition	19
4 Finite Element Error Analysis	22
4.1 Finite Element Discretization	22
4.2 Existence and Uniqueness of the Discrete Solution	23
4.3 Stability of the Finite Element Solution	23
4.4 Error Estimates when $V_{div}^h \not\subset V_{div}$	24
4.5 Approximation Properties and Convergence Rates	27
4.6 Error Estimates for the MINI Element	29
4.7 Pre-Asymptotic Convergence of the Pressure	30
5 Numerical Experiments	32
5.1 Two-Dimensional Test Problem with Smooth Solution	33
5.2 Two-Dimensional Test Problem with Non-Smooth Solution	38
5.3 Two-Dimensional Test Problem with a Complicated Pressure Field	43
5.4 Three-Dimensional Test Problem	47
6 Conclusion	52
Bibliography	53

List of Figures

5.1	Initial grid corresponding to Figure 4.2 in [1].	33
5.2	Velocity field for the smooth two-dimensional test problem visualized on a refined mesh.	34
5.3	Pressure field for the smooth two-dimensional test problem visualized on a refined mesh.	35
5.4	Convergence histories of the velocity and pressure errors for the smooth two-dimensional test problem with $\nu = 1$	37
5.5	Convergence histories of the velocity and pressure errors for the smooth two-dimensional test problem with $\nu = 10^{-6}$	38
5.6	Velocity field for the non-smooth two-dimensional test problem visualized on a refined mesh.	39
5.7	Pressure field for the non-smooth two-dimensional test problem visualised on a refined mesh.	40
5.8	Convergence histories of the velocity and pressure errors for the non-smooth two-dimensional test problem with $\nu = 1$	41
5.9	Convergence histories of the velocity and pressure errors for the non-smooth two-dimensional test problem with $\nu = 10^{-6}$	42
5.10	Approximation of the pressure field for two-dimensional test problem with a complicated pressure field.	44
5.11	Convergence histories of the velocity and pressure errors for the complicated pressure field test problem with $\nu = 1$	45
5.12	Convergence histories of the velocity and pressure errors for the complicated pressure field test problem with $\nu = 10^{-6}$	46
5.13	Grid for $N = 4$ for three-dimensional test problem.	49
5.14	Convergence histories of the velocity and pressure errors for the smooth three-dimensional test problem with $\nu = 1$ and $\alpha = 1$	51

List of Tables

5.1	Degrees of freedom for the smooth two-dimensional test problem.	36
5.2	Discretization errors for the smooth two-dimensional test problem with $\nu = 1$.	36
5.3	Discretization errors for the smooth two-dimensional test problem with $\nu = 10^{-6}$.	37
5.4	Discretization errors for the non-smooth two-dimensional test problem with $\nu = 1$.	40
5.5	Discretization errors for non-smooth two-dimensional test problem with $\nu = 10^{-6}$.	42
5.6	Discretization errors for two-dimensional test problem with a complicated pressure field for $\nu = 1$.	44
5.7	Discretization errors for two-dimensional test problem with a complicated pressure field for $\nu = 10^{-6}$.	46
5.8	Degrees of freedom for the three-dimensional test problem.	49
5.9	Discretization error for the three-dimensional test problem with $\nu = 1$ and $\alpha = 1$.	50

Chapter 1

Introduction

The Navier-Stokes equations describe the motion of incompressible fluids by modeling two physical conservation laws: the conservation of mass and the conservation of linear momentum through a system of partial differential equations. The Navier-Stokes equations essentially represent Newton's second law of motion, which states that the force F acting on an object is the mass m of the object multiplied by its acceleration a , that is, $F = ma$. In the context of fluids dynamics, these equations mainly account for three forces acting on fluid: pressure gradient, viscosity and gravity. The Navier-Stokes equations can be approximated by the Stokes equations for the steady-state situation and the case where the viscosity of the fluid is large. The study of the Navier-Stokes equations is important because incompressible flow problems arise in many scientific and engineering applications, including geophysical flows, biological systems, and microfluidics.

Hence, they are part of many complex models for describing phenomena in nature. However, accurately and efficiently solving the Stokes equations is a central challenge in computational fluid dynamics, as it is usually not possible to find an analytic solution of the Stokes or Navier-Stokes equations. Therefore, we try to approximate the solution. To approximate solutions, one must rely on numerical methods that require the discretizations of the governing equations. The main focus here is on finite element methods (FEM) due to their flexible framework for solving Stokes equations, and also from the mathematical point of view they are understood quite well.

The discretizations of the Stokes equations is a fundamental task in numerical fluid dynamics, with stability and accuracy being key challenges due to the saddle-point structure of the equations. A popular method is the MINI element due to its simplicity and effectiveness. The MINI element is a low-order pair of finite element pairs for discretizing the Stokes equations. The idea of the MINI element is to start with the simplest discretization and then later enrich with a bubble function. This enrichment is inexpensive from a computational point of view and also ensures the satisfaction of the discrete inf-sup (Ladyzhenskaya-Babuška-Brezzi) condition. Therefore, the MINI element is not only theoretically appealing, but also practical for implementation in computational fluid dynamics software.

This thesis presents a detailed survey of the MINI element in the context of the Stokes equations. It aims to:

- Define the MINI element and explain its construction.
- Present a proof of its inf-sup stability.
- Review error estimates in commonly used norms.
- Discuss the phenomenon of higher order error reduction in the pressure approximation.
- Support theoretical results with numerical studies.

The focus is on understanding and clearly presenting the theoretical foundations and numerical behavior of the MINI element. The goal is to offer a self-contained and rigorous yet accessible resource for students like me and anyone interested in finite element methods for incompressible flows.

Chapter 2

The Stokes Equations

The Stokes equations are a simplified form of the Navier–Stokes equations. The Navier–Stokes equations apply to compressible fluids (whose density varies in space and/or time), incompressible fluids (with constant density), viscous fluids (which exhibit internal friction), and steady or unsteady flows. In other words, they apply to all fluids that obey the assumptions of viscosity and fluid continuum.

In general, the Navier–Stokes equations for an incompressible fluid are

$$\begin{aligned}\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u &= f \quad \text{in } \Omega \times (0, T), \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \times (0, T),\end{aligned}\tag{2.0.1}$$

where,

- $t \in (0, T)$: Time interval
- $u(x, t)$: Velocity,
- $p(x, t)$: Pressure,
- ρ : Constant density (because of incompressible fluid case),
- μ : Dynamic viscosity,
- $\nu = \frac{\mu}{\rho}$: Kinematic viscosity,
- f : Body force (can be gravity),
- Δ : Laplacian operator,
- $(u \cdot \nabla)u$: Nonlinear convective term (inertia).

Remark 2.0.1. *The resulting non-dimensionalized equations, with the dimensionless viscosity defined as $\nu = ULRe^{-1}$, (where the same symbol ν may also denote the kinematic viscosity), constitute the fundamental form used in the mathematical analysis and numerical simulation of the incompressible Navier–Stokes equations. In this formulation, U represents a characteristic velocity scale of the flow, such as the mean or inlet velocity, and L denotes a characteristic length scale associated with the physical domain, for instance, the diameter of a pipe, the width of a channel*

or other representative geometric dimension. These scales are employed in the non-dimensionalization process to define the Reynolds number $Re = \frac{UL}{\nu_{phys}}$, where ν_{phys} is the physical kinematic viscosity. For further simplification of the governing equations, one may choose $\nu = 1$ effectively rescaling the problem and normalizing the viscous term.

The Stokes equations are valid for incompressible fluids at low Reynolds numbers. These problems are typically steady-state and linear, as the nonlinear convective term is neglected. Consequently, the mathematical nonlinearity associated with the convective term is removed, and the numerical difficulties that occur when convection dominates over viscosity do not appear in this model. However, the coupling between velocity and pressure remains a challenge.

This motivates us to study the Stokes equations and related important results which can also be used in the analysis of more complicated problems.

2.1 Linearity

The Stokes equations are a simplified case of the Navier–Stokes equations, arising from the absence of the convective term $(u \cdot \nabla)u$, which is nonlinear and characteristic of the Navier–Stokes formulation. When the viscous term $\nu \Delta u$ dominates, the convective effects become negligible. As a result, the Stokes equations are linear; this linearity simplifies both the analytical and numerical treatment of the equations. In many cases, we can consider the flow to be stationary $\partial_t u = 0$. The resulting momentum equation can then be scaled by the Reynolds number. One obtains the following equations called Stokes equations that have to be equipped with appropriate boundary condition.

$$\begin{aligned} -\nu \Delta u + \nabla p &= f & \text{in } \Omega, \\ \nabla \cdot u &= 0 & \text{in } \Omega. \end{aligned} \tag{2.1.1}$$

2.2 The Weak Formulation

The weak (variational) formulation of the Stokes problem leads to a mixed problem with a linear saddle-point structure. Such problems require specific conditions for the well-posedness and stability of the problem. In order to formulate the weak formulation the Stokes equations we require selecting appropriate function spaces for velocity and pressure.

Consider the following formulation of the Stokes equations. For simplicity, we considered the homogeneous Dirichlet boundary condition with $u = 0$ on the boundary Γ :

$$\begin{aligned} -\nu \Delta u + \nabla p &= f & \text{in } \Omega, \\ \nabla \cdot u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma. \end{aligned} \tag{2.2.1}$$

To derive the weak formulation, we begin by multiplying the first equation by a vector-valued test function v , the divergence constraint by a scalar test function q ,

then integrate these equations over the domain Ω , and finally apply integration by parts to transfer the derivatives from the solution to the test functions. The boundary terms vanish due to the homogeneous Dirichlet condition. Given $f \in H^{-1}(\Omega)$, this yields:

$$\begin{aligned} (\nu \nabla u, \nabla v) - (\nabla \cdot v, p) &= \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad \forall v \in H_0^1(\Omega)^d, \\ -(\nabla \cdot u, q) &= 0 \quad \forall q \in L_0^2(\Omega). \end{aligned}$$

Let the velocity space $V = H_0^1(\Omega)^d$, the space of vector-valued functions with square-integrable derivatives that vanish on the boundary, and the pressure space $Q = L_0^2(\Omega)$, the space of square-integrable functions with zero mean. Considering their dual spaces $V' = H^{-1}(\Omega)^d$, consisting of continuous linear functionals on V and $Q' = (L_0^2(\Omega))'$, i.e. the set of bounded linear functionals on $L_0^2(\Omega)$. Since Q is a closed subspace of $L^2(\Omega)$, one can identify Q with $L_0^2(\Omega)$ itself via the Riesz representation. We derive the so-called weak form of the Stokes equations equipped with homogeneous Dirichlet boundary conditions as follows. Find $(u, p) \in V \times Q$:

$$\begin{aligned} (\nu \nabla u, \nabla v) - (\nabla \cdot v, p) &= \langle f, v \rangle_{V', V} \quad \forall v \in V, \\ -(\nabla \cdot u, q) &= 0 \quad \forall q \in Q. \end{aligned} \tag{2.2.2}$$

More generally, one may view the Stokes weak formulation as an instance of a linear saddle-point problem [5]. Where, we introduce V and Q two real Hilbert spaces with inner products $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_Q$. Their duals V' and Q' are defined in the usual way, and the duality pairing is denoted $\langle \cdot, \cdot \rangle_{V', V}$ and $\langle \cdot, \cdot \rangle_{Q', Q}$.

Remark 2.2.1. (Abstract Linear Saddle Point Problem) Let V be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_V$ and Q be another real Hilbert space with inner products $\langle \cdot, \cdot \rangle_Q$. Let $\|\cdot\|_V$ and $\|\cdot\|_Q$ be the respective induced norms of V and Q . Their corresponding dual spaces are given by V' and Q' , with the dual pairing denoted by $\langle \cdot, \cdot \rangle_{V', V}$ and $\langle \cdot, \cdot \rangle_{Q', Q}$. The respective dual norms are defined as

$$\|\phi\|_{V'} = \sup_{v \in V, v \neq 0} \frac{\langle \phi, v \rangle_{V', V}}{\|v\|_V}, \quad \|\phi\|_{Q'} = \sup_{q \in Q, q \neq 0} \frac{\langle \phi, q \rangle_{Q', Q}}{\|q\|_Q}.$$

Given $(f, r) \in V' \times Q'$. The following problem is called a linear saddle point problem. Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= \langle f, v \rangle_{V', V} \quad \forall v \in V, \\ b(u, q) &= \langle r, q \rangle_{Q', Q} \quad \forall q \in Q, \end{aligned} \tag{2.2.3}$$

where,

$$a(\cdot, \cdot) = V \times V \rightarrow \mathbb{R}, \quad b(\cdot, \cdot) = V \times Q \rightarrow \mathbb{R},$$

are two continuous bilinear form with their respective operator norms defined as

$$\|a\| = \sup_{v, w \in V, v, w \neq 0} \frac{a(v, w)}{\|v\|_V \|w\|_V}, \quad \|b\| = \sup_{v \in V, q \in Q, v, q \neq 0} \frac{b(v, q)}{\|v\|_V \|q\|_Q}.$$

In this framework, the Stokes problem is obtained by choosing,

$$a(u, v) = (\nu \nabla u, \nabla v), \quad b(v, q) = -(\nabla \cdot v, q).$$

2.3 Inf-Sup Condition

The inf-sup condition is an important condition that contributes to ensure the uniqueness and stability of the approximated solution. To understand the motivation of the inf-sup condition, we consider a single equation problem associated with the linear saddle-point problem (2.2.1).

Define a manifold $V(r)$ such that every element satisfies :

$$V(r) = \{v \in V : Bv = r\}, \quad V_0 := V(0) = \ker(B).$$

The manifold V_0 is even a subspace of V . From Hilbert space theory, it follows that there is an orthogonal decomposition, with respect to the inner product of V ,

$$V = V_0^\perp \oplus V_0,$$

where $V_0^\perp$ is the orthogonal complement of V_0 . Then the single equation problem is: Find $u \in V(r)$ such that

$$a(u, v) = \langle f, v \rangle_{V', V} \quad v \in V_0. \quad (2.3.1)$$

If $(u, p) \in V \times Q$ is a solution of problem (2.2.1), then $u \in V(r)$. However, it is still necessary to find a condition to ensure that the converse of this statement holds: if $u \in V(r)$ is a solution of (2.3.1), we can find a unique $p \in Q$ such that (u, p) is the unique solution of the linear saddle-point problem (2.2.1).

Remark 2.3.1. A subset of V' is defined as,

$$\tilde{V}' = \{\phi \in V' : \langle \phi, v \rangle_{V', V} = 0 \quad \forall v \in V_0\} \subset V'.$$

This subset, which is even a closed subspace of V' , contains all linear functionals on V that vanishes for all $v \in V_0 = \ker(B)$.

Lemma 2.3.1. (Inf-sup condition) The following properties are equivalent;

(i) There exists a constant $\beta_{is} > 0$ such that

$$\inf_{q \in Q, q \neq 0} \sup_{v \in V, v \neq 0} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta_{is}. \quad (2.3.2)$$

(ii) The operator B' is an isomorphism Q onto $\tilde{V}'$ and

$$\|B'q\|_{V'} \geq \beta_{is} \|q\|_Q \quad \forall q \in Q. \quad (2.3.3)$$

(iii) The operator B is an isomorphism $V_0^\perp$ onto Q' and

$$\|Bv\|_{Q'} \geq \beta_{is} \|v\|_V \quad \forall v \in V_0^\perp. \quad (2.3.4)$$

Proof: Proof can be found in the book Girault and Raviart, [2]. ■

2.4 Existence and Uniqueness of the Weak Solution

The linear saddle-point problem (2.2.3) can be written in an equivalent form using operators instead of bilinear forms, and we obtain the transformed problem as: Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} Au + B'p &= f \quad \text{in } V', \\ Bu &= r \quad \text{in } Q'. \end{aligned} \quad (2.4.1)$$

Lemma 2.4.1. *Assume that the bilinear form $a(\cdot, \cdot)$ is V_0 -elliptic, i.e., there is a constant $\alpha > 0$ such that*

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V_0.$$

Then the problem (2.4.1) is well posed if and only if the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition (2.3.2).

Proof: The proof is referred to Lemma 3.16 of the book "Finite Element Methods for Incompressible Flow Problems" by Volker John, [1]. ■

Theorem 2.4.1. (Inf-sup condition for V and Q) *The spaces V and Q satisfy the inf-sup condition (2.3.2) that is there exists a $\beta_{is} > 0$ such that*

$$\inf_{q \in Q, q \neq 0} \sup_{v \in V, v \neq 0} \frac{(\nabla \cdot v, q)}{\|v\|_V \|q\|_Q} \geq \beta_{is}. \quad (2.4.2)$$

Proof: The proof refers to Theorem 3.46 of the book "Finite Element Methods for Incompressible Flow Problems" by Volker John, [1]. ■

2.5 Stability of the Solution of the Stokes Equations

Theorem 2.5.1. (Existence and uniqueness of the solution of Stokes equations) *Let Ω be a bounded domain in $\mathbb{R}$, $d \in \{2, 3\}$, with a Lipschitz continuous boundary Γ and let $f \in H^{-1}(\Omega)$. Then there exists a unique pair $(u, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ that solves (2.2.2).*

Proof: The bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition. In addition to this,

$$a(v, v) = \nu \|v\|_{H^1(\Omega)}^2 = \nu \|v\|_V^2 \quad \forall v \in V \quad \text{with } b(v, q) = 0 \quad \forall q \in L_0^2(\Omega). \quad (2.5.1)$$

Thus, $a(\cdot, \cdot)$ is coercive on $\ker b$. Then by Lemma 2.4.1 the statement of this theorem follows directly. ■

Lemma 2.5.1. (Stability of the solution) *Let the condition of Theorem 2.5.1 be given. Then the solution (u, p) of the Stokes equations (2.2.2) depends continuously on the right-hand side,*

$$\|u\|_V = \|\nabla u\|_{L^2(\Omega)} \leq \frac{1}{\nu} \|f\|_{H^{-1}(\Omega)}, \quad (2.5.2)$$

$$\|p\|_Q = \|p\|_{L^2(\Omega)} \leq \frac{2}{\beta_{is}} \|f\|_{H^{-1}(\Omega)}. \quad (2.5.3)$$

If $f \in L^2(\Omega)$, then it holds,

$$\|u\|_V \leq \frac{C}{\nu} \|P_{helm} f\|_{L^2(\Omega)}, \quad (2.5.4)$$

where the constant C comes from Poincaré inequality.

Proof: Consider the Stokes problem (2.4.1) and define a manifold of V such that it contains all elements which fulfill the second equation of problem (2.4.1). The kernel of the operator B , $\ker(B) = V_0 := V(0)$ is important. This kernel is the space of weakly divergence-free functions in V ,

$$V_0 = V_{div} = \{v \in V : (\nabla \cdot v, q) = 0 \quad \forall q \in Q\}. \quad (2.5.5)$$

Note that for $v \in V_{div}$, the divergence vanishes in the L^2 -sense, that is, $\|\nabla \cdot v\|_{L^2(\Omega)} = 0$, which implies $(\nabla \cdot v)(x) = 0$ almost everywhere in Ω . Using this $V_0 = V_{div}$ we reformulate problem (2.2.2), which also corresponds to problem (2.2.3) as: Find $u \in V_{div}$ such that,

$$(\nu \nabla u, \nabla v) = \langle f, v \rangle_{V', V} \quad \forall v \in V_{div}. \quad (2.5.6)$$

Inserting $v = u$ as a test function in (2.5.6) gives,

$$\begin{aligned} (\nabla u, \nabla u) &= \langle f, u \rangle_{V', V}, \\ \Rightarrow \nu \|\nabla u\|_{L^2(\Omega)}^2 &= \langle f, u \rangle_{V', V}. \end{aligned}$$

Then the dual estimate yields the following,

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq \|f\|_{H^{-1}(\Omega)} \|\nabla u\|_{L^2(\Omega)}.$$

Thus,

$$\nu \|\nabla u\|_{L^2(\Omega)}^2 \leq \|f\|_{H^{-1}(\Omega)} \|\nabla u\|_{L^2(\Omega)}.$$

If $\|\nabla u\|_{L^2(\Omega)} = 0$, the velocity statement is valid directly. Otherwise, division with $\|\nabla u\|_{L^2(\Omega)}$ leads to,

$$\|\nabla u\| \leq \frac{1}{\nu} \|f\|_{H^{-1}(\Omega)}.$$

For the pressure estimate, we make use of the inf-sup condition (2.4.2) of Theorem 2.4.1, the problem (2.2.2), the estimate of dual pairing and the Cauchy-Schwarz inequality,

$$\begin{aligned}
 \|p\|_{L^2(\Omega)} &\leq \frac{1}{\beta_{is}} \sup_{v \in V \setminus \{0\}} \frac{-(\nabla \cdot v, p)}{\|\nabla v\|_{L^2(\Omega)}} \\
 &= \frac{1}{\beta_{is}} \sup_{v \in V \setminus \{0\}} \frac{\langle f, v \rangle_{V', V} - (\nu \nabla u, \nabla v)}{\|\nabla v\|_{L^2(\Omega)}} \\
 &\leq \frac{1}{\beta_{is}} \sup_{v \in V \setminus \{0\}} \frac{\|f\|_{H^{-1}(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \nu \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}}{\|\nabla v\|_{L^2(\Omega)}} \\
 &= \frac{1}{\beta_{is}} (\|f\|_{H^{-1}(\Omega)} + \nu \|\nabla u\|_{L^2(\Omega)}) \\
 &\leq \frac{1}{\beta_{is}} (\|f\|_{H^{-1}(\Omega)} + \nu \cdot \frac{1}{\nu} \|f\|_{H^{-1}(\Omega)}) \\
 &\leq \frac{2}{\beta_{is}} \|f\|_{H^{-1}(\Omega)}.
 \end{aligned}$$

Here, the pressure bound (2.5.3) in the last inequality is obtained by using the velocity stability estimate (2.5.2).

Let $f \in L^2(\Omega)$, then the application of the Helmholtz decomposition and the Helmholtz projection gives

$$f = P_{helm}f + \nabla r.$$

Then by utilizing integration by parts, $u \in V_{div}$, the Cauchy-Schwarz inequality and the Poincaré inequality we get,

$$\begin{aligned}
 \nu \|\nabla u\|_{L^2(\Omega)}^2 &= (P_{helm}f, u) + (\nabla r, u) = (P_{helm}f, u) \\
 &\leq \|P_{helm}f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \|P_{helm}f\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)},
 \end{aligned}$$

dividing $\nu \|\nabla u\|_{L^2(\Omega)}$ both sides,

$$\|\nabla u\| \leq C \|P_{helm}f\|_{L^2(\Omega)}.$$

Finally, we can prove the required statement. ■

2.6 Inf-sup (LBB) Condition

Ladyzhenskaya–Babuška–Brezzi (LBB) condition is the stability condition for mixed variational problems. For problems like the Stokes equations, it ensures that the variational formulation of such problems is well-posed. It also ensures that the choice of finite-element spaces leads to stable discretizations. In this section, we will look at the proof of this condition and some tools required for the proof.

Corollary 2.6.1. (*Each Pressure is the Divergence of a Velocity Field*) For each $q \in Q$ there is a unique $v \in V_{div}^\perp \subset V$ such that

$$\nabla \cdot v = q \quad \text{and} \quad \|q\|_Q \leq \|v\|_V, \quad \|v\|_V \leq C \|q\|_Q, \quad (2.6.1)$$

with C independent of v and q .

Proof: The existence of a unique $v \in V_{div}^\perp$ with $\nabla \cdot v = q$ follows from the Lemma 3.43 in [1], which states the isomorphism of the divergence operator. The operator div is an isomorphism from $V_{div}^\perp$ onto Q . Further, for the functions from $V = H_0^1(\Omega)$ it holds,

$$\|\nabla \cdot v\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega),$$

with this, we get,

$$\|q\|_Q = \|\nabla \cdot v\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)} = \|v\|_V.$$

The inverse map of the divergence is also an isomorphism. In particular, it is bounded. Hence, there is a $C > 0$ such that

$$\|v\|_V = \|div^{-1}q\|_V \leq C\|q\|_Q \text{ for all } q \in Q \text{ and all } v \in V_{div}^\perp. \quad \blacksquare$$

Corollary 2.6.2. (Estimating the Gradient by the Divergence for Functions from $V_{div}^\perp$) For all $v \in V_{div}^\perp$ it holds that,

$$\|v\|_V \leq \frac{1}{\beta_{is}} \|\nabla \cdot v\|_{L^2(\Omega)}. \quad (2.6.2)$$

Proof: By Corollary (2.6.1) and the continuous inf-sup condition (cf. Theorem 3.46 in [1]) we have,

$$\beta_{is} = \inf_{0 \neq q \in Q} \sup_{0 \neq w \in V} \frac{(\nabla \cdot w, q)}{\|w\|_V \|q\|_{L^2(\Omega)}} > 0,$$

which implies that the operator $\nabla \cdot : V_{div}^\perp \rightarrow Q$ is continuously invertible and that its inverse satisfies

$$\|v\|_V \leq \frac{1}{\beta_{is}} \|q\|_{L^2(\Omega)}.$$

Since $q = \nabla \cdot v$, we conclude that

$$\|v\|_V \leq \frac{1}{\beta_{is}} \|\nabla \cdot v\|_{L^2(\Omega)}.$$

This proves (2.6.2). \blacksquare

Lemma 2.6.1. (Ladyzhenskaya–Babuška–Brezzi Condition) It holds

$$\begin{aligned} & \inf_{\substack{(w,r) \in V \times Q \\ (w,r) \neq (0,0)}} \sup_{\substack{(v,q) \in V \times Q \\ (v,q) \neq (0,0)}} \frac{(\nabla w, \nabla v) - (\nabla \cdot v, r) + (\nabla \cdot v, q)}{(\|\nabla w\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega)})(\|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)})} \\ & \geq \beta_{is, Bab} > 0 \end{aligned} \quad (2.6.3)$$

with,

$$\frac{1}{\beta_{is,Bab}} = \left(1 + \frac{2}{\beta_{is}}\right) + \frac{1}{\beta_{is}} \left(1 + \frac{1}{\beta_{is}}\right).$$

Proof: Let $(w, r) \in V \times Q$ be arbitrary. Consider the operator associated with the variational form of the Stokes equations to be $f \in V'$, $g \in Q$ and define for all $(v, q) \in V \times Q$,

$$(\nabla w, \nabla v) - (\nabla \cdot v, r) = \langle f, v \rangle_{V', V}, \quad (\nabla \cdot w, q) = (g, q). \quad (2.6.4)$$

We begin the proof by decomposing w into a divergence-free part and an orthogonal complement. Let w_{div} be the orthogonal projection of w onto V_{div} with respect to the V -inner product. Then $(w - w_{div}) \in (V_{div})^\perp$ and from Corollary (2.6.1) $\nabla \cdot (w - w_{div}) \in Q$. Using (2.6.4) with the test function in V_{div} gives $\nabla \cdot w_{div} = 0$.

Now, computing the estimate for the divergence part $\nabla \cdot (w - w_{div})$,

$$\begin{aligned} (\nabla \cdot w, \nabla \cdot (w - w_{div})) &= (\nabla \cdot (w - w_{div}) \cdot \nabla \cdot (w - w_{div})) \\ &= \|\nabla \cdot (w - w_{div})\|_{L^2(\Omega)}^2. \end{aligned}$$

But by (2.6.4) the LHS can be written as,

$$\|\nabla \cdot (w - w_{div})\|_{L^2(\Omega)}^2 = (g, \nabla \cdot (w - w_{div})).$$

Hence, by the Cauchy-Schwarz inequality,

$$|(g, \nabla \cdot (w - w_{div}))| \leq \|g\|_{L^2(\Omega)} \|\nabla \cdot (w - w_{div})\|_{L^2(\Omega)},$$

and by applying the continuous LBB estimate (2.6.2), one obtains,

$$\|\nabla(w - w_{div})\|_{L^2(\Omega)} \leq \frac{1}{\beta_{is}} \|\nabla \cdot (w - w_{div})\|_{L^2(\Omega)}$$

We get the bound,

$$\|\nabla(w - w_{div})\|_{L^2(\Omega)} \leq \frac{1}{\beta_{is}} \|g\|_{L^2(\Omega)}.$$

Next, we derive the estimate for the divergence-free part w_{div} . Since $\nabla \cdot w_{div} = 0$, we choose $v = w_{div}$ in (2.6.4), which gives,

$$(\nabla w, \nabla w_{div}) - (\nabla \cdot w_{div}, r) = (\nabla w, \nabla w_{div}) = \langle f, w_{div} \rangle_{V', V}.$$

From the orthogonality, it follows that,

$$\begin{aligned} (\nabla w, \nabla w_{div}) &= (\nabla(w - w_{div}), \nabla w_{div}) + (\nabla w_{div}, \nabla w_{div}) \\ &= (\nabla w_{div}, \nabla w_{div}) = \|\nabla w_{div}\|_{L^2(\Omega)}^2, \end{aligned}$$

such that we obtain for the dual pairing,

$$\|\nabla w_{div}\|_{L^2(\Omega)}^2 = \langle f, w_{div} \rangle_{V', V} \leq \|f\|_{H^{-1}(\Omega)} \|\nabla w_{div}\|_{L^2(\Omega)}.$$

Hence, the bound,

$$\|\nabla w_{div}\|_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}.$$

2. The Stokes Equations

To obtain a bound on $\|\nabla w\|$, we apply the triangle inequality and combine the two bounds estimated above

$$\begin{aligned}\|\nabla w\|_{L^2(\Omega)} &\leq \|\nabla(w - w_{div})\|_{L^2(\Omega)} + \|\nabla w_{div}\|_{L^2(\Omega)} \\ &\leq \frac{1}{\beta_{is}}\|g\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)}.\end{aligned}\tag{2.6.5}$$

Using the same technique as the stability estimate for the pressure in Lemma 2.5.1 gives the relation,

$$\|r\|_{L^2(\Omega)} \leq \frac{1}{\beta_{is}}(\|f\|_{H^{-1}(\Omega)} + \|\nabla w\|_{L^2(\Omega)}).$$

Substituting (2.6.5) into this relation yields,

$$\begin{aligned}\|r\|_{L^2(\Omega)} &\leq \frac{1}{\beta_{is}}\left(\|f\|_{H^{-1}(\Omega)} + \frac{1}{\beta_{is}}\|g\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)}\right) \\ &\leq \frac{2}{\beta_{is}}\|f\|_{H^{-1}(\Omega)} + \frac{1}{\beta_{is}^2}\|g\|_{L^2(\Omega)}.\end{aligned}\tag{2.6.6}$$

By definition,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in V \setminus \{0\}} \frac{\langle f, v \rangle_{V', V}}{\|\nabla v\|_{L^2(\Omega)}} = \sup_{v \in V \setminus \{0\}} \frac{(\nabla w, \nabla v) - (\nabla \cdot v, r)}{\|\nabla v\|_{L^2(\Omega)}}.$$

Also,

$$\begin{aligned}\|g\|_{L^2(\Omega)} &= \sup_{q \in L^2(\Omega) \setminus \{0\}} \frac{(g, q)}{\|q\|_{L^2(\Omega)}} = \sup_{q \in L^2(\Omega) \setminus \{0\}} \frac{(\nabla \cdot w, q)}{\|q\|_{L^2(\Omega)}} \\ &= \sup_{q \in Q \setminus \{0\}} \frac{(\nabla \cdot w, q)}{\|q\|_{L^2(\Omega)}}.\end{aligned}$$

Define the combined supremum,

$$S := \sup_{(v, q) \neq (0, 0)} \frac{(\nabla w, \nabla v) - (\nabla \cdot v, r) + (\nabla \cdot w, q)}{\|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)}}.$$

- Take any $v \in V$ and set $q = 0$. Then

$$\frac{(\nabla w, \nabla v) - (\nabla \cdot v, r)}{\|v\|} = \frac{(\nabla w, \nabla v) - (\nabla \cdot v, r) + (\nabla \cdot w, 0)}{\|\nabla v\| + 0} \leq S.$$

- Take any $q \in Q$ and set $v = 0$. Then

$$\frac{(\nabla w, q)}{\|q\|} = \frac{(\nabla w, \nabla 0) - (\nabla \cdot 0, r) + (\nabla \cdot w, q)}{0 + \|q\|} \leq S.$$

Adding (2.6.5) and (2.6.6) leads to

$$\begin{aligned}
& \|\nabla w\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega)} \\
& \leq \left(\frac{1}{\beta_{is}} \|g\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)} \right) + \left(\frac{2}{\beta_{is}} \|f\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Omega)} \right) \\
& = \left(1 + \frac{2}{\beta_{is}} \right) \|f\|_{H^{-1}(\Omega)} + \left(\frac{1}{\beta_{is}} + \frac{1}{\beta_{is}^2} \right) \|g\|_{L^2(\Omega)}.
\end{aligned}$$

Since $\|f\|_{H^{-1}(\Omega)} \leq S$ and $\|g\|_{H^{-1}(\Omega)} \leq S$ we can get the bound as

$$\begin{aligned}
\|\nabla w\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega)} & \leq \left[\left(1 + \frac{2}{\beta_{is}} \right) + \left(\frac{1}{\beta_{is}} + \frac{1}{\beta_{is}^2} \right) \right] S \\
& \leq \left[\left(1 + \frac{2}{\beta_{is}} \right) + \frac{1}{\beta_{is}} \left(1 + \frac{1}{\beta_{is}^2} \right) \right] S.
\end{aligned}$$

Finally, by dividing LHS by first factor on RHS and taking the infimum over non-zero (w, r) to obtain the required inf-sup statement,

$$\begin{aligned}
\inf_{\substack{(w,r) \in V \times Q \\ (w,r) \neq (0,0)}} \sup_{\substack{(v,q) \in V \times Q \\ (v,q) \neq (0,0)}} & = \frac{(\nabla w, \nabla v) - (\nabla \cdot v, r) + (\nabla \cdot v, q)}{(\|\nabla w\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega)})(\|\nabla v\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)})} \\
& \geq \beta_{is, Bab} > 0,
\end{aligned}$$

with

$$\frac{1}{\beta_{is, Bab}} = \left(1 + \frac{2}{\beta_{is}} \right) + \frac{1}{\beta_{is}} \left(1 + \frac{1}{\beta_{is}} \right). \quad \blacksquare$$

Chapter 3

Discretization with the MINI Element

3.1 MINI Element

The construction of the MINI element begins with standard finite element spaces for velocity and pressure. Then the velocity space is enriched with local bubble functions. These bubble functions are polynomial functions that vanish on the element boundaries and are non-zero only in the interior of each element. By making only slight changes to the basic discretization and bubble enrichment, the MINI element not only ensures satisfaction of the discrete inf-sup (or Ladyzhenskaya–Babuška–Brezzi) condition but also increases the approximation power of the velocity space.

The MINI element [6] is defined on the simplicial grid and is given by

$$V^h = P_1 \bigoplus V_{bub}^h, \quad Q^h = P_1, \quad (3.1.1)$$

where V_{bub}^h is a space consisting of local bubble functions.

$$V_{bub}^h = \{v_{bub}^h : \text{supp}(V_{bub}^h) = K, v_{bub}^h|_K = \alpha \prod_{i=1}^{d+1} \lambda_i, K \in \mathcal{T}^h, \alpha \in \mathbb{R}\},$$

where λ_i are the barycentric coordinates of the simplex K . It follows that

$$v_{bub}^h|_K \in P_{d+1}(K) \cap H_0^1(K).$$

This is the lowest-order conforming inf-sup stable pair of finite element spaces.

Lemma 3.1.1. (*Properties of bubble functions*)

Let $K \in \mathcal{T}^h$ be a simplex and let

$$v_{bub}^h(x) = \prod_{i=1}^{d+1} \lambda_i(x), \quad x \in K,$$

be a bubble function on K . Then the following estimates hold

$$\|v_{bub}^h\|_{L^2(K)} \leq Ch_K^d, \quad (3.1.2)$$

$$\|\nabla v_{bub}^h\|_{L^2(K)} \leq Ch_K^{(d-2)/2}, \quad (3.1.3)$$

$$\int_K v_{bub}^h(x) dx \geq C|K|, \quad (3.1.4)$$

where the constants are independent of K .

Proof: The estimate (3.1.2) follows directly by Holder's inequality.

$$\|v_{bub}^h\|_{L^2(K)} \leq \|v_{bub}^h\|_{L^\infty(K)} \|1\|_{L^1(K)} \leq \|v_{bub}^h\|_{L^\infty(K)} |K|.$$

The maximum value of v_{bub}^h is uniformly bounded by a constant depending only on the reference element, since v_{bub}^h is a polynomial of degree $d+1$ defined on a simplex with diameter h_K , that is,

$$\|v_{bub}^h\|_{L^\infty(K)} \leq C.$$

Using the volume scaling $|K| \sim h_K^d$, we obtain,

$$\|v_{bub}^h\|_{L^\infty(K)} \leq Ch_K^{d/2}.$$

For the estimate (3.1.3), let $F : \hat{K} \rightarrow K$ be an affine mapping from the reference simplex $\hat{K}$ to the physical simplex K ,

$$x = F(\xi) = B\xi + b,$$

where $B \in \mathbb{R}^{d \times d}$ is the Jacobi matrix of the mapping.

Now $\hat{v}_{bub}^h(x) = \prod_{i=1}^{d+1} \hat{\lambda}_i(x)$ is the reference bubble, then using the chain rule gives,

$$\nabla_x v_{bub}^h(x) = B^{-T} \nabla_\xi \hat{v}_{bub}^h(\xi).$$

Applying the affine transformation and using the change of variables gives,

$$\begin{aligned} \|\nabla v_{bub}^h\|_{L^2(K)}^2 &= \int_K (\nabla v_{bub}^h \cdot \nabla v_{bub}^h)(x) dx \\ &= |\det(B)| \int_{\hat{K}} B^{-T} \nabla_\xi \hat{v}_{bub}^h(\xi) \cdot B^{-T} \nabla_\xi \hat{v}_{bub}^h(\xi) d\xi. \end{aligned}$$

The last integral depends only on the reference element and can be considered constant. Also we know that for a regular shape family of meshes,

$$|\det(B)| \sim h_K^d, \quad \|B^{-T}\|_2 \sim h_K^{-1}.$$

Therefore,

$$\|\nabla v_{bub}^h\|_{L^2(K)}^2 \leq Ch_K^d h_K^{-2}$$

this implies,

$$\|\nabla v_{bub}^h\|_{L^2(K)} \leq Ch_K^{(d-2)/2}.$$

Lastly, for the estimate (3.1.4), we use the bubble functions $v_{bub}^h = \prod_{i=1}^{d+1} \lambda_i(x)$ of degree $d+1$ in the interior of K which vanish on the boundary.

To evaluate its integral, consider a quadrature rule on K with positive weights $\omega_i > 0$ and nodes $x_i \in K$. There are quadrature rules for polynomials of degree $2 \leq k \leq 7$ for $d = 2$ and of degree $1 \leq k \leq 5$ for $d = 3$, exact for polynomials of degree up to $2(d+1)$:

$$\int_K v_{bub}^h(x) dx = |K| \sum_{i=1}^{N_0} \omega_i v_{bub}^h(x_i).$$

Since $v_{bub}^h(x_i) \neq 0$, for all interior nodes and attains its maximum at the barycenter of K

$$\|v_{bub}^h\|_{L^\infty(K)} = \begin{cases} \frac{1}{27}, & \text{if } d = 2, \\ \frac{1}{216}, & \text{if } d = 3. \end{cases}$$

It follows that

$$\begin{aligned} \left| \int_K v_{bub}^h(x) dx \right| &= |K| \left| \sum_{i=1}^{N_0} \omega_i v_{bub}^h(x_i) \right| = |K| \sum_{i=1}^{N_0} \omega_i v_{bub}^h(x_i) \\ &\geq |K| \min_{i=1, \dots, N_0} v_{bub}^h(x_i) \sum_{i=1}^{N_0} \omega_i = C|K|, \end{aligned}$$

where $C > 0$ is a constant depending only on the dimension d . ■

In all these three scaling estimates, the constants C are dependent solely on the reference bubble integrals and the shape-regularity constants. Consequently, they are uniform across a shape regular family of meshes. This uniformity allows us to treat the bubble as a controlled, local enrichment. Its size and gradient behavior are predictable, independent of the element's position, and solely determined by its diameter h_k and the mesh's shape regularity. Given that the MINI element enriches the linear velocity space by including one interior bubble per element, the bubble's integral and gradient must not be too small. This requirement is essential to ensure that the enriched velocity space can be properly paired with the pressure space to satisfy the discrete inf-sup (Ladyzhenskaya–Babuška–Brezzi) condition. Further, the bubble function is used not only to construct local test functions called Fortin operators and local stability estimates, but also to derive error estimates that exploit local scaling of approximation and stability constants.

Consider the weak formulation (2.2.2) of the Stokes equations where $V = H_0^1(\Omega)$ and $Q = L_0^2(\Omega)$ equipped with appropriate norms. However, these are infinite-dimensional spaces, and we now wish to replace these spaces with finite-dimensional spaces in order to apply the finite-dimensional methods. For incompressible flow problems, the pair of velocity-pressure finite element spaces is denoted by V^h containing vector-valued functions and Q^h is possibly intersected with $L_0^2(\Omega)$ depending on the boundary

condition. In particular, spaces V^h and Q^h now have to satisfy the following inf-sup condition, called the discrete inf-sup condition.

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{v^h \in V^h \setminus \{0\}} \frac{b^h(v^h, q^h)}{\|v^h\|_{V^h} \|q^h\|_{Q^h}} \geq \beta_{is}^h > 0. \quad (3.1.5)$$

For confirming finite element space, it is possible to check the discrete inf-sup condition (3.1.5) with the help of the continuous inf-sup condition (2.4.2) we discussed earlier.

3.2 Fortin Operator

To understand the connection between the continuous inf-sup condition (2.4.2) and the discrete inf-sup condition (3.1.5) we look at the Fortin operator. Later, its generalization can be made to some non-confirming spaces.

Lemma 3.2.1. Fortin Criterion for Checking the Discrete Inf-Sup Condition [7] Let V , Q and $b(\cdot, \cdot)$ fulfill the assumptions of Remark 2.2.1 and let the inf-sup condition (2.4.2) be satisfied. Consider conforming spaces $V^h \subset V$ and $Q^h \subset Q$. Then, V^h and Q^h satisfy the discrete inf-sup condition (3.1.5) if and only if there exists a constant $\gamma^h > 0$, which is indecent of h , such that for all $v \in V$ there is an element $P_{For}^h v \in V^h$ with

$$b(v, q^h) = b(P_{For}^h v, q^h) \quad \forall q^h \in Q^h \quad \text{and} \quad \|P_{For}^h v\|_V \leq \gamma^h \|v\|_V. \quad (3.2.1)$$

Proof: The proof is referred follows from book "Finite Element Methods for Incompressible Flow Problems" by Volker John. [1]

$\Leftarrow$ Assume that (3.2.1) is true.

Let $q^h \in Q^h$ be arbitrary. From $\text{span}\{P_{For}^h\} \subseteq V^h$, and now using (3.2.1) and (2.4.2) it follows that

$$\begin{aligned} \sup_{v^h \in V^h \setminus \{0\}} \frac{b(v^h, q^h)}{\|v^h\|_V} &\geq \sup_{v \in V \setminus \{0\}} \frac{b(P_{For}^h v, q^h)}{\|P_{For}^h v\|_V} = \sup_{v \in V \setminus \{0\}} \frac{b(v, q^h)}{\|P_{For}^h v\|_V} \\ &\geq \sup_{v \in V \setminus \{0\}} \frac{b(v, q^h)}{\gamma^h \|v\|_V} \geq \frac{\beta_{is}}{\gamma^h} \|q^h\|_{Q^h}. \end{aligned}$$

This inequality is nothing but the discrete inf-sup condition (3.1.5) with $\beta_{is}^h = \beta_{is}/\gamma^h$.

$\Rightarrow$ Assume that (3.1.5) is true.

Consider the restriction of $b(\cdot, \cdot)$ from $V \times Q$ to $V \times Q^h$. This restriction defines a continuous linear operator,

$$\tilde{B} \in \mathcal{L}(V, (Q^h)'), \quad \langle \tilde{B}v, q \rangle_{(Q^h)', Q^h} = b(v, q^h),$$

for all $v \in V$ with $\tilde{B}v \in (Q^h)'$ by definition. Since the discrete inf-sup condition (3.1.5) holds, it follows from (2.3.3) that the operator B^h associated with the bilinear form $b(\cdot, \cdot)$,

$$B^h : V^h \rightarrow (Q^h)', \quad \langle B^h v^h, q^h \rangle_{(Q^h)', Q^h} = b^h(v^h, q^h),$$

the discrete divergence operator, is an isomorphism from $(V_{div}^h)^\perp$ onto $(Q^h)'$. In particular, B^h is surjective. Since $\tilde{B}v \in (Q^h)'$, there must be an $\tilde{v}^h$ from $(V_{div}^h)^\perp$ such that

$$\langle B^h \tilde{v}^h, q^h \rangle_{(Q^h)', Q^h} = \langle \tilde{B}v, q^h \rangle_{(Q^h)', Q^h} \quad \forall q^h \in Q^h.$$

Consequently, for all $v^h \in V^h$ whose projection into $(V_{div}^h)^\perp$ is equal to $\tilde{v}^h$, it holds.

$$\langle B^h v^h, q^h \rangle_{(Q^h)', Q^h} = \langle \tilde{B}v, q^h \rangle_{(Q^h)', Q^h} \quad \forall q^h \in Q^h. \quad (3.2.2)$$

Choosing one of these elements as $P_{For}^h v$. Then, (3.2.2) is equivalent to

$$b(P_{For}^h v, q^h) = b(v, q^h) \quad \forall q \in Q^h.$$

$$b(v, q) = - \int_{\Omega} (\nabla \cdot v) q dx = -(\nabla \cdot v, q) \quad \forall v \in V, q \in Q,$$

is bounded by

$$|b(v, q)| = \|v\|_V \|q\|_Q,$$

and consequently it is continuous. In addition, $\|b\| = 1$.

With these relations, it follows from (2.3.3), the definition of the norm in Q^h , the definition of the norm of $b(\cdot, \cdot)$ in Remark 2.2.1 and the estimate of this norm that,

$$\begin{aligned} \|P_{For}^h v\|_V &\leq \frac{1}{\beta_{is}^h} \|B^h(P_{For}^h v)\|_Q = \frac{1}{\beta_{is}^h} \sup_{q^h \in Q^h \setminus \{0\}} \frac{\langle B^h(P_{For}^h v), q^h \rangle_{(Q^h)', Q}}{\|q^h\|_Q} \\ &= \frac{1}{\beta_{is}^h} \sup_{q^h \in Q^h \setminus \{0\}} \frac{b^h(P_{For}^h v, q^h)}{\|q^h\|_Q} = \frac{1}{\beta_{is}^h} \sup_{q^h \in Q^h \setminus \{0\}} \frac{b^h(v, q^h)}{\|q^h\|_Q} \\ &\leq \frac{1}{\beta_{is}^h} \sup_{q^h \in Q^h \setminus \{0\}} \frac{\|b\| \|v\|_V \|q^h\|_Q}{\|q^h\|_Q} = \frac{\|b\|}{\beta_{is}^h} \|v\|_V = \gamma^h \|v\|_V. \end{aligned}$$

Since v was arbitrarily chosen, (3.2.1) is proved. ■

Generalization of Fortin Criterion (3.2.1): Rewrite the first part of the Fortin criterion (3.2.1) in the form

$$- \int_K \nabla \cdot (v - P_{For}^h v) q^h dx = 0 \quad \forall v \in V, \forall q^h \in Q^h.$$

Then for Confirming finite element spaces and a continuous finite element pressure space using integration by parts, it follows that,

$$\int_{\Omega} (v - P_{For}^h v) \cdot \nabla q^h dx = 0 \quad \forall v \in V, \forall q^h \in Q^h. \quad (3.2.3)$$

Further, replace the global criterion (3.2.3) with the set of local criteria

$$\int_K (v - P_{For}^h v) \cdot \nabla q^h dx = 0 \quad \forall v \in V, \forall q^h \in Q^h, \forall K \in \mathcal{T}^h. \quad (3.2.4)$$

For the construction of the Fortin operator. We can note here that (3.2.4) induces (3.2.3) but the inverse is not true.

Construction of the Fortin operator: The Fortin operator is based on the Cl  ment interpolation operator [8] P_{Cle}^h , with the modification to preserve homogeneous Dirichlet boundary conditions. Consider a quasi-uniform family of triangulations. Then, the number of mesh cells in the set ω_K is uniformly bounded from above and we get the global estimate,

$$\sum_{K \in \mathcal{T}^h} h_K^{-2} \|v - P_{Cle}^h v\|_{L^2(K)}^2 \leq C \|v\|_{H^1(\Omega)}^2 \quad \forall v \in H^1(\Omega), \quad (3.2.5)$$

$$\sum_{K \in \mathcal{T}^h} \|\nabla(v - P_{Cle}^h v)\|_{L^2(K)}^2 \leq C \|v\|_{H^1(\Omega)}^2 \quad \forall v \in H^1(\Omega). \quad (3.2.6)$$

Now by applying the triangle inequality we get the stability estimate,

$$\begin{aligned} & \sum_{K \in \mathcal{T}^h} \|\nabla P_{Cle}^h v\|_{L^2(K)}^2 \\ & \leq 2 \left(\sum_{K \in \mathcal{T}^h} \|\nabla(v - P_{Cle}^h v)\|_{L^2(K)}^2 + \sum_{K \in \mathcal{T}^h} \|\nabla v\|_{L^2(K)}^2 \right) \leq C \|v\|_{H^1(\Omega)}^2. \end{aligned} \quad (3.2.7)$$

Now, the Fortin operator is defined by

$$P_{For}^h v(x) = P_{Cle}^h v(x) + \alpha_K v_{bub}^h(x), \quad (3.2.8)$$

with

$$\alpha_K = \frac{\int_K (v - P_{For}^h v)(x) dx}{\int_K v_{bub}^h(x) dx}. \quad (3.2.9)$$

3.3 Discrete Inf-Sup Condition

Theorem 3.3.1. (Discrete Inf-Sup Condition for the MINI Element) Consider a quasi-uniform family of triangulations. Then the operator

$$P_{For}^h v(x) = P_{Cle}^h v(x) + \alpha_K v_{bub}^h(x),$$

with,

$$\alpha_K = \frac{\int_K (v - P_{For}^h v)(x) dx}{\int_K v_{bub}^h(x) dx},$$

is a Fortin operator. Hence, the MINI element (3.1.1) satisfies the discrete inf-sup condition (3.1.5).

Proof: We must verify that the Fortin operator P_{For}^h both conditions in (3.2.1). To do this, we consider the sufficient local condition:

$$\int_K (v - P_{For}^h v)(x) dx = 0 \quad \forall v \in V, \quad \forall K \in \mathcal{T}^h. \quad (3.3.1)$$

Substituting into (3.2.8) and (3.2.9) into (3.3.1). Then for any element K we have,

$$\begin{aligned} \int_K (v - P_{For}^h v)(x) dx &= \int_K (v - P_{Cle}^h v - \alpha_K v_{bub}^h)(x) dx \\ &= \int_K (v - P_{Cle}^h v)(x) dx - \alpha_K \int_K v_{bub}^h(x) dx. \end{aligned}$$

By inserting the definition of α_K , the two terms cancel exactly,

$$\begin{aligned} &= \int_K (v - P_{Cle}^h v)(x) dx - \frac{\int_K (v - P_{Cle}^h v)(x) dx}{\int_K v_{bub}^h(x) dx} \int_K v_{bub}^h(x) dx \\ &= 0. \end{aligned}$$

Hence, the local mean value condition $b(v - P_{For}^h v, q^h) = 0, \forall q^h \in Q^h$, in (3.3.1) is satisfied.

Now we want to verify that P_{For}^h is stable in H^1 , the stability property in (3.1.5), that is: $\|P_{For}^h v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega)}$.

Using the triangle inequality and linearity of the gradient, we have, for each element K ,

$$\|\nabla P_{For}^h v\|_{L^2(K)} \leq \|\nabla P_{Cle}^h v\|_{L^2(K)} + \|\alpha_K\|_2 \|\nabla v_{bub}^h\|_{L^2(K)}. \quad (3.3.2)$$

Here, α_K is a vector-valued constant, and $\|\alpha_K\|_2$ denotes its Euclidean norm.

Next, we want to estimate the bubble contribution. Therefore, we estimate the second term on the right-hand side of (3.3.2) and by the definition of α_K ,

$$\|\alpha_K\|_2 = \frac{\|\int_K (v - P_{For}^h v)(x) dx\|_2}{|\int_K v_{bub}^h(x) dx|}.$$

Using the Cauchy–Schwarz inequality, the fact that $|K| = Ch_K^d$ for quasi-uniform meshes, and the scaling properties of the bubble functions (3.1.3) – (3.1.4), we obtain,

$$\begin{aligned} \|\alpha_K\|_2 \|\nabla v_{bub}^h\|_{L^2(K)} &\leq C \frac{\|\int_K (v - P_{Cle}^h v)(x) dx\|_2}{|\int_K v_{bub}^h(x) dx|} h_K^{(d-2)/2} \\ &\leq C \frac{\|v - P_{Cle}^h v\|_{L^2(K)} |K|^{1/2}}{|K|} h_K^{(d-2)/2} \\ &\leq C \|v - P_{Cle}^h v\|_{L^2(K)} h_K^{d/2-1-d/2} \\ &= Ch_K^{-1} \|v - P_{Cle}^h v\|_{L^2(K)}. \end{aligned}$$

Thus, the bubble correction term scales like $h_K^{-1} \|v - P_{Cle}^h v\|_{L^2(K)}$.

Substituting this bound into (3.3.2) gives

$$\|\nabla P_{For}^h v\|_{L^2(K)} \leq \|\nabla P_{Cle}^h v\|_{L^2(K)} + Ch_K^{-1} \|v - P_{Cle}^h v\|_{L^2(K)}.$$

Squaring both sides and applying Young's inequality leads to

$$\|P_{For}^h v\|_V^2 \leq C \left(\|\nabla P_{Cle}^h v\|_{L^2(K)}^2 + h_K^{-2} \|v - P_{Cle}^h v\|_{L^2(K)}^2 \right).$$

Summing over all elements $K \in \mathcal{T}^h$ gives,

$$\|P_{For}^h v\|_V^2 \leq C \left(\sum_{K \in \mathcal{T}^h} \|\nabla P_{Cle}^h v\|_{L^2(K)}^2 + h_K^{-2} \|v - P_{Cle}^h v\|_{L^2(K)}^2 \right).$$

Using the standard properties of the Clément interpolant and the Poincaré inequality,

$$\|\nabla P_{Cle}^h v\|_{L^2(\Omega)} = C \|\nabla v\|_{L^2(\Omega)}, \quad \|v - \nabla P_{Cle}^h v\|_{L^2(K)} = Ch_K \|\nabla v\|_{L^2(\omega_K)},$$

we obtain,

$$\|P_{For}^h v\|_V \leq C \|v\|_V.$$

This verifies the stability condition in [\(3.2.1\)](#).

Therefore, P_{For}^h is a Fortin operator and consequently the MINI element satisfies the discrete inf-sup condition [\(3.1.5\)](#). ■

Chapter 4

Finite Element Error Analysis

Finite element error analysis provides a rigorous framework for quantifying how accurately a numerical approximation reproduces the exact solution of a given partial differential equation. Since the finite element method replaces a continuous problem by a discrete one defined on a mesh of elements, some level of approximation error is inevitable. Error analysis quantifies this discrepancy, allowing engineers and analysts to

- evaluate the reliability of their simulations,
- identify regions where the mesh should be refined (for example, near sharp corners, steep gradients, or stress concentrations), and
- understand the rate at which the numerical solution converges to the exact (weak) solution as the mesh is refined.

Consequently, finite element error analysis not only builds confidence in computational predictions, but also guides efficient and targeted improvements of both the mesh and the numerical model.

We recall the weak formulation of the Stokes problem. Given $f \in H^{-1}(\Omega)$, find $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\begin{aligned}(\nu \nabla u, \nabla v) - (\nabla \cdot v, p) &= \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad \forall v \in H_0^1(\Omega)^d, \\ -(\nabla \cdot u, q) &= 0 \quad \forall q \in L_0^2(\Omega).\end{aligned}\tag{4.0.1}$$

If (u, p) is sufficiently smooth solution of the strong Stokes problem (2.2.2), then it is also a solution of the weak formulation (4.0.1). Conversely, starting from the strong formulation (2.2.2), the weak formulation (4.0.1) is obtained by testing (2.2.2) with functions of the form $\{(v, 0)\}$ and $\{(0, q)\}$ as test functions.

4.1 Finite Element Discretization

To carry out a finite element error analysis, we introduce finite-dimensional subspaces. Let $V^h \subset H_0^1(\Omega)$ be a finite-dimensional velocity space and let $Q^h \subset L_0^2(\Omega)$ be a finite element pressure space. The finite element discretization of the Stokes problem (4.0.1) then reads: Find $(u^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} a^h(u^h, v^h) + b^h(v^h, p^h) &= \sum_{K \in \mathcal{T}^h} \int_K (f v^h)(x) dx & \forall v^h \in V^h \\ b^h(u^h, q^h) &= 0 & \forall q^h \in Q^h. \end{aligned} \quad (4.1.1)$$

Here, the discrete bilinear forms are defined by

$$a^h(v^h, w^h) = \sum_{K \in \mathcal{T}^h} (\nabla v^h, \nabla w^h)_K, \quad b^h(v^h, q^h) = - \sum_{K \in \mathcal{T}^h} (\nabla \cdot v^h, q^h)_K. \quad (4.1.2)$$

We assume that the right-hand side f is sufficiently smooth such that the linear functional on the right-hand side of the momentum equation in (4.1.1) is well defined. This discrete formulation with finite element spaces serves as the foundation for analyzing the stability, solvability, and convergence of the finite element approximation.

4.2 Existence and Uniqueness of the Discrete Solution

Theorem 4.2.1. (*Existence and Uniqueness of a Solution of the Finite Element Stokes equations*) Let $(a^h(\cdot, \cdot))^{1/2}$ define a norm in V^h and let the discrete inf-sup condition (3.1.5) hold. Then, (4.1.1) has a unique solution.

Proof: The argument is analogous to the proof of Theorem (2.5.1). Since $(a^h(\cdot, \cdot))^{1/2}$ defines a norm in V^h , the bilinear form a^h is then V^h -elliptic and, in particular, elliptic on the discrete divergence-free subspace V_{div}^h . Together with the discrete inf-sup condition, this ensures existence and uniqueness of the discrete solution. ■

Corollary 4.2.1. (*Unique solvability of the Finite Element Problem*) Let V^h and Q^h be conforming finite element spaces that satisfy the discrete inf-sup condition (3.1.5). Then the finite element problem (4.1.1) has a unique solution.

Proof: In (2.5.1) we saw that $V^h \subset V$, the norm induced by $(a^h(\cdot, \cdot))^{1/2}$ restricts to a norm on V^h and based on Theorem (4.2.1) it is sufficient to show that $(a(\cdot, \cdot))^{1/2}$ defines a norm in V^h . ■

4.3 Stability of the Finite Element Solution

Lemma 4.3.1. Let $V^h \times Q^h$ be a pair of inf-sup stable finite element spaces. Then, the solution of (4.1.1) fulfills

$$\|\nabla u^h\|_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}, \quad \|p^h\|_{L^2(\Omega)} \leq \frac{2}{\beta_{is}^h} \|f\|_{H^{-1}(\Omega)}.$$

Proof: The proof follows the same steps as the stability proof for the continuous problem Lemma (2.5.1). ■

4.4 Error Estimates when $V_{div}^h \not\subset V_{div}$

Theorem 4.4.1. (Finite Element Error Estimate for the $L^2(\Omega)$ Norm of the Gradient of the Velocity) Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polyhedral domain with Lipschitz boundary. Let $(u, p) \in V \times Q$ be a unique solution of the Stokes problem (4.0.1). Assume that this problem is discretized with the inf-sup stable conforming finite element spaces $V^h \times Q^h$ and denote by $u^h \in V_{div}^h$ the velocity solution. Then, the following error estimate holds,

$$\|\nabla(u - u^h)\|_{L^2(\Omega)} \leq 2 \inf_{v^h \in V_{div}^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} + \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}. \quad (4.4.1)$$

Proof: The proof proceeds by deriving an error equation and estimating its terms. The discrete velocity can be characterized as the solution of the following problem: Find $u^h \in V_{div}^h$ such that

$$a^h(u^h, v^h) = (\nabla u^h, \nabla v^h) = \langle f, v^h \rangle_{V', V} \quad \forall v^h \in V_{div}^h. \quad (4.4.2)$$

We may test the continuous Stokes problem (2.2.2), which is equivalent to (4.0.1), with functions in V_{div}^h and subtract (4.4.2). This yields the error equation

$$(\nabla(u - u^h), \nabla v^h) - (\nabla \cdot v^h, p) = 0 \quad \forall v^h \in V_{div}^h. \quad (4.4.3)$$

Since, in general, $V_{div}^h \not\subset V_{div}$, the pressure does not vanish in the error equation. We now modify the pressure terms so that an approximation term with respect to the pressure is obtained. Observing that $(\nabla \cdot v^h, q^h) = 0$ for all $q^h \in Q^h$ leads to

$$(\nabla(u - u^h), \nabla v^h) - (\nabla \cdot v^h, p - q^h) = 0 \quad \forall v^h \in V^h, \quad \forall q^h \in Q^h. \quad (4.4.4)$$

Decomposing the error using an arbitrary interpolant $I^h u \in V_{div}^h$, we write the approximation error for the velocity as

$$u - u^h = (u - I^h u) - (u^h - I^h u) = \eta - \phi^h.$$

Hence, η is just an approximation error which depends only on the finite element space V_{div}^h but not on the considered equation. The goal is to estimate $\phi^h \in V_{div}^h$, using approximation errors as well. The decomposition is inserted in (4.4.4) and the test functions ϕ^h is chosen. It follows that

$$\|\nabla \phi^h\|_{L^2(\Omega)}^2 = (\nabla \phi^h, \nabla \phi^h) = (\nabla \eta, \nabla \phi^h) - (\nabla \cdot \phi^h, p - q^h) \quad \forall q^h \in Q^h. \quad (4.4.5)$$

Now, the terms on the right-hand side are estimated using the Cauchy-Schwarz inequality

$$|(\nabla \eta, \nabla \phi^h)| \leq \|\nabla \eta\|_{L^2(\Omega)} \|\nabla \phi^h\|_{L^2(\Omega)}$$

and, using in addition

$$\begin{aligned} |-(\nabla \phi^h, p - q^h)| &\leq \|p - q^h\|_{L^2(\Omega)} \|\nabla \cdot \phi^h\|_{L^2(\Omega)} \\ &\leq \|p - q^h\|_{L^2(\Omega)} \|\nabla \phi^h\|_{L^2(\Omega)}. \end{aligned} \quad (4.4.6)$$

Inserting these estimates in (4.4.5) and dividing by $\|\nabla \phi^h\|_{L^2(\Omega)} \neq 0$ yields the following

$$\|\nabla \phi^h\|_{L^2(\Omega)} \leq \|\nabla \eta\|_{L^2(\Omega)} + \|p - q^h\|_{L^2(\Omega)}.$$

In the case that $\|\nabla \phi^h\|_{L^2(\Omega)} = 0$, this estimate trivially holds.

With the triangle inequality it follows that

$$\begin{aligned} \|\nabla(u - u^h)\|_{L^2(\Omega)} &\leq \|\nabla \phi^h\|_{L^2(\Omega)} + \|\nabla \eta\|_{L^2(\Omega)} \\ &\leq 2\|\nabla \eta\|_{L^2(\Omega)} + \|p - q^h\|_{L^2(\Omega)} \end{aligned}$$

for all $I^h u \in V_{div}^h$, and for all $q^h \in Q^h$, from what (4.4.1) follows. ■

Corollary 4.4.1. (*Finite Element Error Estimate for the $L^2(\Omega)$ Norm of the Divergence of the Velocity*) Let the assumptions of Theorem 4.4.1 be fulfilled then

$$\|\nabla \cdot u^h\|_{L^2(\Omega)} \leq 2 \inf_{v^h \in V_{div}^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} + \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}. \quad (4.4.7)$$

Proof: Using the relation $\nabla \cdot u = 0$ we have

$$\|\nabla \cdot u^h\|_{L^2(\Omega)} = \|\nabla \cdot (u - u^h)\|_{L^2(\Omega)}$$

Using the continuity of the divergence operator, there exists a constant $C > 0$ such that

$$\|\nabla \cdot v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega)^d.$$

Hence,

$$\|\nabla \cdot u_h\|_{L^2(\Omega)} \leq C \|\nabla(u - u_h)\|_{L^2(\Omega)}.$$

The result then follows from (4.4.1) ■

Theorem 4.4.2. (*Finite Element Error Estimate for the $L^2(\Omega)$ Norm of the Pressure*) Let the assumption of Theorem 4.4.1 be satisfied. Then the following error estimate holds

$$\|p - p^h\|_{L^2(\Omega)} \leq \frac{2}{\beta_{is}^h} \inf_{v^h \in V_{div}^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} + \left(1 + \frac{2}{\beta_{is}^h}\right) \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}. \quad (4.4.8)$$

Proof: Let $q^h \in Q^h$ be arbitrary, then one gets with triangle inequality

$$\|p - p^h\|_{L^2(\Omega)} \leq \|p - q^h\|_{L^2(\Omega)} + \|p^h - q^h\|_{L^2(\Omega)}.$$

To estimate the second term, we employ the discrete inf-sup condition (3.1.5)

$$\beta_{is}^h \leq \inf_{q^h \in Q^h} \sup_{v^h \in V^h \setminus \{0\}} \frac{b^h(v^h, q^h)}{\|v^h\|_{V^h} \|q^h\|_{Q^h}}.$$

It follows that

$$\|p^h - q^h\|_{L^2(\Omega)} \leq \frac{1}{\beta_{is}^h} \sup_{v^h \in V^h \setminus \{0\}} \frac{b(v^h, p - q^h)}{\|\nabla v^h\|_{L^2(\Omega)}}.$$

Next, we use the continuous and discrete momentum equations of the Stokes problem. For arbitrary $v^h \in V^h$, subtracting the discrete momentum equation from the continuous one yields

$$b(v^h, p^h - q^h) = -a(u^h, v^h) + \langle f, v^h \rangle_{V', V} - b(v^h, q^h) = a(u - u^h) + b(v^h, p - q^h).$$

Inserting this identity into the inf-sup estimate and applying the Cauchy-Schwarz inequality together with the continuity of the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, we obtain

$$\begin{aligned} \frac{b(v^h, p^h - q^h)}{\|\nabla v^h\|_{L^2(\Omega)}} &\leq \frac{|a(u - u^h, v^h)|}{\|\nabla v^h\|_{L^2(\Omega)}} + \frac{|b(v^h, p - q^h)|}{\|\nabla v^h\|_{L^2(\Omega)}} \\ &\leq \|\nabla(u - u^h)\|_{L^2(\Omega)} + \|p - q^h\|_{L^2(\Omega)}. \end{aligned}$$

Hence,

$$\|p^h - q^h\|_{L^2(\Omega)} \leq \frac{1}{\beta_{is}^h} (\|\nabla(u - u^h)\|_{L^2(\Omega)} + \|p - q^h\|_{L^2(\Omega)}), \quad \forall q^h \in Q^h.$$

Combining this estimate with the initial triangle inequality yields

$$\|p - p^h\|_{L^2(\Omega)} \leq \left(1 + \frac{1}{\beta_{is}^h}\right) \|p - q^h\|_{L^2(\Omega)} + \frac{1}{\beta_{is}^h} \|\nabla(u - u^h)\|_{L^2(\Omega)}.$$

Finally, we employ the velocity error estimate derived in Theorem 4.4.1

$$\|\nabla(u - u^h)\|_{L^2(\Omega)} \leq 2 \inf_{v^h \in V_{div}^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} + \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}.$$

Inserting this bound and taking the infimum over $q^h \in Q^h$ yields

$$\|p - p^h\|_{L^2(\Omega)} \leq \frac{2}{\beta_{is}^h} \inf_{v^h \in V_{div}^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} + \left(1 + \frac{2}{\beta_{is}^h}\right) \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)},$$

which proves the asserted estimate. ■

For the remainder of this chapter, we assume that the exact solution satisfies the regularity conditions

$$u \in H^2(\Omega)^d, \quad p \in H^1(\Omega),$$

unless otherwise stated. These assumptions ensure that the approximation estimates derived below are well defined and allow explicit convergence rates to be obtained for the finite element approximation.

Remark 4.4.1. (Pressure convergence rates in $L^2(\Omega)$) Let the assumptions of Theorem 4.4.2 be satisfied. Assume, in addition, that the exact solution fulfills $u \in H^{k+1}(\Omega)^d$, $p \in H^{l+1}(\Omega)$ and that the finite element spaces $V^h \subset V$, $Q^h \subset Q$ satisfy the approximation properties

$$\inf_{v^h \in V_{div}^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} \leq Ch^k |u|_{H^{k+1}(\Omega)},$$

$$\inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)} \leq Ch^{l+1} |p|_{H^{l+1}(\Omega)}.$$

Then there exists a constant $C > 0$, independent of h , such that

$$\|p - p^h\|_{L^2(\Omega)} \leq C(h^k |u|_{H^{k+1}(\Omega)} + h^{l+1} |p|_{H^{l+1}(\Omega)}).$$

This estimate is optimal in the asymptotic sense.

4.5 Approximation Properties and Convergence Rates

The error estimates derived above are formulated in terms of best approximation errors in the discrete spaces V_{div}^h and Q^h . In order to obtain explicit convergence rates with respect to the mesh size h , we recall standard approximation results for conforming finite element spaces. Here and in the following, the constant $C > 0$ denotes a generic constant that is independent of the mesh size h , but may depend on the domain Ω , the polynomial degrees of the finite element spaces, and the shape regularity of the family of meshes.

Assume that the velocity space V^h contains all piecewise polynomial functions of degree $k \geq 1$, and that the pressure space Q^h contains all piecewise polynomials of degree $m \geq 0$. If the exact solution satisfies

$$u \in H^{k+1}(\Omega)^d, \quad p \in H^{m+1}(\Omega),$$

then there exist interpolation operators $I^h : V \rightarrow V^h$ and $\Pi^h : Q \rightarrow Q^h$ satisfying the standard interpolation estimates

$$\|\nabla(u - I^h u)\|_{L^2(\Omega)} \leq Ch^k \|u\|_{H^{k+1}(\Omega)},$$

$$\|p - \Pi^h p\|_{L^2(\Omega)} \leq Ch^{m+1} \|p\|_{H^{m+1}(\Omega)}.$$

By the definition of the best approximation, these estimates imply

$$\inf_{v^h \in V_{div}^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} \leq Ch^k \|u\|_{H^{k+1}(\Omega)}, \quad (4.5.1)$$

and

$$\inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)} \leq Ch^{m+1} \|p\|_{H^{m+1}(\Omega)}. \quad (4.5.2)$$

Combining these best approximation estimates with the velocity error bound (4.4.1) yields the convergence rate

$$\|\nabla(u - u^h)\|_{L^2(\Omega)} \leq C(h^k \|u\|_{H^{k+1}(\Omega)} + h^{m+1} \|p\|_{H^{m+1}(\Omega)}). \quad (4.5.3)$$

Similarly, the pressure error estimate (4.4.8) implies

$$\|p - p^h\|_{L^2(\Omega)} \leq C(h^k \|u\|_{H^{k+1}(\Omega)} + h^{m+1} \|p\|_{H^{m+1}(\Omega)}). \quad (4.5.4)$$

These estimates show that the convergence of the finite element solution is governed by both the approximation properties of the discrete spaces and the regularity of the exact solution.

Remark 4.5.1. (*Inf-Sup Stability and Pressure Effects*) It is important to notice that the velocity error estimate (4.4.1) contains a contribution from the pressure approximation error. This term arises because, in general, the discrete divergence-free space V_{div}^h is not a subspace of the continuous divergence-free space V_{div} . As a consequence, inaccuracies in the pressure approximation may affect the quality of the velocity approximation.

Finite element methods for which $V_{div}^h \subset V_{div}$ yield exactly divergence-free discrete velocities and eliminate this pressure-dependent term in the velocity error estimate. Such methods are commonly referred to as pressure-robust schemes.

Remark 4.5.2. (*Dependence on the viscosity and Reynolds number*) Consider the scaled Stokes problem

$$-\nu \Delta u + \nabla p = f,$$

Dividing the momentum equation by ν leads to a formulation of the same type as the standard Stokes problem, but with a rescaled pressure and right-hand side. Consequently, the finite element error analysis yields velocity error bounds containing terms proportional to

$$\frac{1}{\nu} \|p\|_{H^{m+1}(\Omega)}.$$

Hence, for small viscosities (equivalently, large Reynolds numbers $Re = UL/\nu$), the pressure-dependent contribution may dominate the velocity error. In contrast, the pressure error estimate does not exhibit the same scaling with respect to ν . In (4.5.4), all pressure-dependent terms are initially multiplied by $1/\nu$. Multiplying the entire

estimate by ν then eliminates this factor on the left-hand side, while introducing a factor ν in front of the velocity approximation term. Therefore, the pressure error does not contain an inverse dependence on the viscosity. Consequently, the accuracy of the velocity approximation may deteriorate as the Reynolds number increases, even for smooth solutions.

Remark 4.5.3. (Convergence behavior of the pressure) The estimate (4.5.4) predicts convergence of the pressure in the L^2 -norm with the minimum of $\mathcal{O}(h^k)$ and $\mathcal{O}(h^{m+1})$, provided sufficient regularity of the exact solution. In practice, however, numerical experiments may exhibit deviations from the asymptotic regime on coarse meshes or, in some cases, improved convergence behavior. This aspect will be further investigated in the numerical experiments.

4.6 Error Estimates for the MINI Element

We now specialize the abstract error analysis to the MINI finite element for the stationary Stokes problem in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions. The MINI element consists of a velocity space $V_h \subset H_0^1(\Omega)^d$ formed by enriching continuous piecewise linear functions with one cubic bubble function per element, and a pressure space $Q_h = P_1(T_h) \cap L_0^2(\Omega)$. This pair is known to satisfy a mesh independent discrete inf-sup condition (3.1.5), ensuring stability of the discrete formulation.

For sufficiently smooth solutions $u \in H^2(\Omega)^d$ and $p \in H^1(\Omega)$, the approximation properties of the MINI element yield

$$\begin{aligned} \inf_{v_h \in V_h} \|\nabla(u - v_h)\|_{L^2(\Omega)} &\leq Ch \|u\|_{H^2(\Omega)}, \\ \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega)} &\leq Ch \|p\|_{H^1(\Omega)}. \end{aligned}$$

Combining these estimates with the general velocity and pressure error bounds (4.5.3) and (4.5.4), we obtain the convergence result

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq Ch(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}).$$

Thus, the MINI finite element yields optimal first order convergence of the velocity in the H^1 -norm and of the pressure in the L^2 -norm, provided that the exact solution is sufficiently smooth. In order to verify these results and to illustrate their practical implications, we present some numerical experiments in the next chapter.

Since the discrete divergence-free velocity space associated with the MINI element is not a subspace of the continuous divergence-free space, the method is not pressure-robust: the velocity error may still be influenced by inaccuracies in the pressure approximation, as noted above.

Although the pressure error estimate derived in Theorem 4.4.2 is optimal, numerical experiments often reveal that the pressure approximation in the $L^2(\Omega)$ -norm converges at a higher rate on relatively coarse meshes than predicted by the asymptotic theory. This behavior cannot be explained by the classical inf-sup theory alone and indicates the presence of a pre-asymptotic regime with enhanced accuracy, which we investigate in the following section through a Schur-complement reformulation of the discrete Stokes system.

4.7 Pre-Asymptotic Convergence of the Pressure

A theoretical explanation for the experimentally observed pre-asymptotic pressure super-convergence was recently proposed by Pacheco and Steinbach [3], through a Schur complement analysis of the discrete Stokes system.

Consider first the continuous saddle-point formulation of the Stokes problem,

$$\begin{pmatrix} A & -B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Eliminating the velocity variable formally yields the continuous Schur-complement equation

$$Sp := (CA^{-1}B + D)p = g - CA^{-1}f =: F.$$

The corresponding weak formulation reads: find $p \in L^2(\Omega)$ such that

$$(Sp, q) = (F, q) \quad \forall q \in L^2(\Omega).$$

In theory, this problem may be approximated directly using a finite element pressure space Q^h , yielding approximation solution $p^h \in Q^h$. Standard finite element approximation theory then gives

$$\|p - p^h\|_{L^2(\Omega)} \leq Ch^{l+1}|p|_{H^{l+1}(\Omega)}, \quad (4.7.1)$$

where l denotes the polynomial degree of the pressure space Q^h , and $C > 0$ is independent of the mesh size h .

In practice, however, one does not solve the exact Schur-complement problem, since the continuous operator S is not available. Instead, the finite element discretization of the Stokes system produces the algebraic system

$$\begin{pmatrix} A_h & -B_h \\ C_h & D_h \end{pmatrix} \begin{pmatrix} u^h \\ p^h \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

and consequently, only an approximate discrete Schur-complement operator is available,

$$\tilde{S}^h = C_h A_h^{-1} B_h + D_h.$$

The associated reduced problem is: find $\tilde{p}^h \in Q^h$ such that

$$(\tilde{S}_h \tilde{p}^h, q_h) = (\tilde{F}, q_h) \quad \forall q_h \in Q^h.$$

Under the assumption $\tilde{F} = F$, Pacheco and Steinbach prove in [3, Theorem 3.1] the estimate

$$\|p - \tilde{p}^h\|_{L^2(\Omega)} \leq \alpha h^{m+1} |p|_{H^{l+1}(\Omega)} + \beta h^k |w|_{H^{k+1}(\Omega)}, \quad (4.7.2)$$

where $w = A^{-1}Bp$ and k denotes the polynomial degree of the velocity approximation space.

For the MINI element one has $k = l = 1$. In this case, estimate (4.7.2) alone predicts only first order pressure convergence because the second term dominates asymptotically.

However, the analysis in [3] shows that on coarse meshes the situation can be substantially better. Applying the triangle inequality together with the approximation estimate (4.7.1) yields

$$\frac{1}{2} \|p - \tilde{p}^h\|_{L^2(\Omega)}^2 \leq \|p - p^h\|_{L^2(\Omega)}^2 + \|p^h - \tilde{p}^h\|_{L^2(\Omega)}^2. \quad (4.7.3)$$

Further calculations, using (4.7.2) and assumption (13) from [3], one obtains

$$\frac{1}{2} \|p - \tilde{p}^h\|_{L^2(\Omega)}^2 \leq Ch^{2(l+1)} |p|_{H^{l+1}(\Omega)}^2 + C_1 h^{2l+1} |p|_{H^{m+1}(\Omega)}^2 + C_2 \|p - \tilde{p}^h\|_{L^2(\Omega)}^2 \quad (4.7.4)$$

with $C_2 \leq 1/2$, allowing the final term to be absorbed into the left-hand side. This yields an effective convergence rate of order $l + \frac{1}{2}$ explaining why numerical experiments often display pressure convergence rates between 1.5 and 2 during the initial refinement levels.

The improved convergence behaviour is, however, only pre-asymptotic. The crucial assumption (13) from [3] takes the form

$$C_3 |p|_{H^{l+1}(\Omega)} (C_4 h^{l+1} + C_5 \|A^{-1}B\| h^l) < \frac{1}{2} \|p - \tilde{p}^h\|_{L^2(\Omega)}.$$

Since the left-hand side behaves like h^l and the right-hand side like $h^{l+1/2}$, this assumption eventually fails as $h \rightarrow 0$, and the classical asymptotic convergence rate predicted by standard inf-sup theory is recovered on sufficiently fine meshes.

The analysis also relies on sufficient regularity of the auxiliary quantity $w = A^{-1}Bp$. If the inverse velocity operator does not provide full H^2 -regularity, for example in domains with re-entrant corners or non-smooth data, then $w \notin H^{k+1}(\Omega)$ and the enhanced convergence mechanism may disappear.

The Schur-complement framework therefore provides an alternative interpretation of the pressure approximation and offers a theoretical explanation for the enhanced pressure convergence frequently observed on coarse meshes for stabilized mixed finite element discretizations such as the MINI element.

Chapter 5

Numerical Experiments

In this chapter, we present numerical experiments designed to examine the theoretical error estimates developed in Chapter 4 for finite element discretizations of the stationary incompressible flow problem. The experiments focus on the convergence behaviour of the velocity and pressure approximations of the MINI finite element, in particular, on the influence of the viscosity parameter and the regularity of the pressure solution on the discretization error. We consider several manufactured solution problems for which the exact velocity and pressure fields are known. This allows the discretization errors to be computed directly and provides a controlled setting in which the theoretical convergence estimates can be examined numerically.

The experiments are organized progressively. We first consider a smooth two-dimensional manufactured solution in order to establish the basic convergence behaviour of the MINI finite element discretization and to examine the influence of the viscosity parameter. We then consider a second manufactured solution satisfying the assumptions of the refined pressure analysis in Chapter 4 in order to investigate enhanced pressure convergence. Finally, we introduce a pressure field containing a sharp localized transition in order to study how the structure and regularity of the pressure affect the observed pressure convergence.

Throughout this chapter, we consider the stationary Stokes problem 2.2.1

$$\begin{aligned} -\nu \Delta u + \nabla p &= f & \text{in } \Omega, \\ \nabla \cdot u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

Its weak formulation reads: Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= \langle f, v \rangle_{V', V} & \forall v \in V, \\ b(u, q) &= 0 & \forall q \in Q, \end{aligned}$$

where

$$V := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega),$$

and the bilinear forms are defined as

$$a(u, v) := (\nu \nabla u, \nabla v), \quad b(v, q) := -(\nabla \cdot v, q).$$

The corresponding finite element approximation reads: Find $(u_h, p_h) \in V_h \times Q_h$ satisfying

$$\begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= (f, v_h) & \forall v_h \in V_h, \\ b(u_h, q_h) &= 0 & \forall q_h \in Q_h. \end{aligned} \tag{5.0.1}$$

We measure the errors in the norms

$$\|\nabla(u - u_h)\|_{L^2(\Omega)}, \quad \|u - u_h\|_{L^2(\Omega)}, \quad \|p - p_h\|_{L^2(\Omega)}.$$

Unless stated otherwise, all numerical experiments in this chapter are performed on the same initial triangular mesh shown in Figure 5.1 corresponding to Figure 4.2 from [1]. Uniform refinement is applied by subdividing each triangle into four congruent triangles at every refinement level. This provides a consistent mesh hierarchy across the different manufactured solution problems and allows differences in the numerical results to be attributed to the manufactured solution or the viscosity parameter rather than to changes in the mesh.

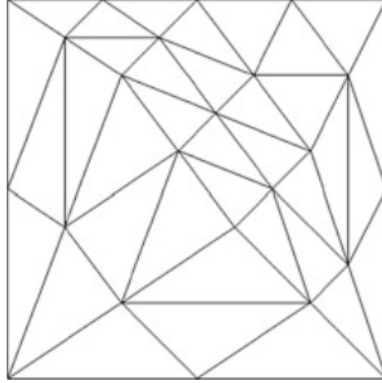


Figure 5.1: Initial grid corresponding to Figure 4.2 in [1].

5.1 Two-Dimensional Test Problem with Smooth Solution

We first consider a two-dimensional test problem on the unit square $\Omega = (0, 1)^2$ with a smooth manufactured solution. This example serves as a reference problem for examining the convergence behaviour of the MINI finite element discretization. It is also used to determine how reducing the viscosity parameter ν affects the magnitude and experimental convergence rates of the velocity and pressure errors.

Exact Solution

The velocity field is prescribed as

$$u(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix} = 1000 \begin{pmatrix} x^2(1-x)^4 y^2(1-y)(3-5y) \\ -2x(1-x)^3(1-3x)y^3(1-y)^2 \end{pmatrix}.$$

The pressure is given by

$$p(x, y) = \pi^2(xy^3 \cos(2\pi x^2 y) - x^2 y \sin(2\pi xy)) + \frac{1}{8}.$$

The velocity field is divergence-free and vanishes on the boundary of the domain. The constant in the pressure is chosen such that the normalization condition

$$\int_{\Omega} p \, dx = 0$$

is satisfied.

The corresponding right-hand side f is computed analytically from the governing Stokes equations according to

$$f = -\nu \Delta u + \nabla p,$$

Consequently, the pair (u, p) is the exact solution of the continuous problem.

The numerical experiments start from the initial triangular mesh shown in Figure [5.1](#). Figure [5.2](#) and [5.3](#) illustrate the spatial structure of the prescribed velocity and pressure fields. These visualizations are included to characterize the manufactured solution qualitatively, the convergence properties of the finite element approximation are assessed quantitatively from the discretization errors under successive mesh refinement.

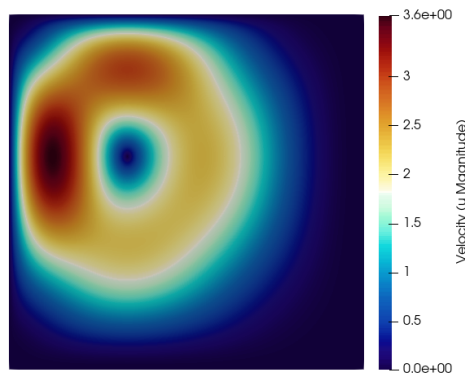


Figure 5.2: Velocity field for the smooth two-dimensional test problem visualized on a refined mesh.

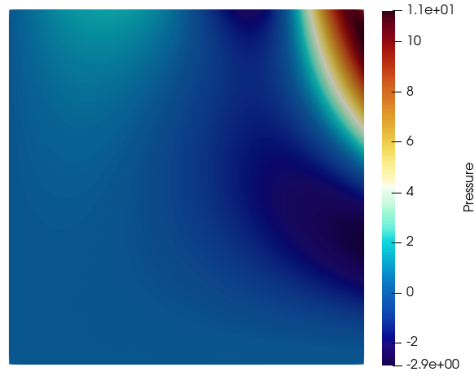


Figure 5.3: Pressure field for the smooth two-dimensional test problem visualized on a refined mesh.

For the pressure visualization in Figure 5.3, a sufficiently fine visualization mesh is used. The colour bar represents the pressure variable p directly and is positioned adjacent to the pressure field for clarity.

Purpose of the Test Problem

The theoretical error estimates derived in Chapter 4 contain viscosity-dependent constants and therefore the magnitude of the velocity error may increase as ν decreases. By repeating the same manufactured solution on the same mesh hierarchy for two viscosity values, the influence of ν can be separated from changes caused by the solution or the discretization mesh.

The experiment also provides a first numerical examination of the pressure approximation. In particular, we compare the observed pressure convergence rate with the classical first order behaviour and investigate whether an enhanced pressure convergence rate is visible for this smooth manufactured solution.

Numerical Results for $\nu = 1$

The numerical experiment uses the initial triangular mesh in Figure 5.1 and the described uniform refinement strategy, where at each refinement level, each triangle is subdivided into four congruent triangles, producing a systematic sequence of uniformly refined meshes.

The numbers of velocity, pressure and total degrees of freedom associated with each refinement level are reported in Table 5.1.

Table 5.1: Degrees of freedom for the smooth two-dimensional test problem.

Level	Velocity DOFs	Pressure DOFs	Total DOFs
1	120	24	144
2	454	83	537
3	1770	309	2079
4	6994	1193	8187
5	27810	4689	32499
6	110914	18593	129507
7	443010	74049	517059
8	1770754	295553	2066307

The systematic increase in the number of degrees of freedom confirms the uniform refinement of the mesh hierarchy used for the convergence study.

Table 5.2 shows a systematic reduction of all three discretization errors under mesh refinement. The experimental convergence rate of the $H^1(\Omega)$ -seminorm velocity error approaches order one, while the $L^2(\Omega)$ -velocity error approaches second order convergence. These results agree with the expected behaviour for the smooth manufactured velocity solution.

Table 5.2: Discretization errors for the smooth two-dimensional test problem with $\nu = 1$.

Level	$\ \nabla(u - u_h)\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ p - p_h\ _{L^2(\Omega)}$	Rate
1	8.67761	–	0.59785	–	5.82071	–
2	4.48828	0.95	0.17208	1.80	1.66396	1.81
3	2.35767	0.93	0.04886	1.82	0.58804	1.50
4	1.19349	0.98	0.01259	1.96	0.25410	1.21
5	0.59764	1.00	0.00316	1.99	0.10170	1.32
6	0.29868	1.00	0.00079	2.00	0.03824	1.41
7	0.14926	1.00	0.00020	1.98	0.01386	1.46
8	0.07460	1.00	4.91702e-05	2.02	0.00495	1.49

The pressure error also decreases systematically under mesh refinement. Its experimental convergence rate approaches approximately 1.5 on the finer meshes. This behaviour is higher than the classical first order estimate and provides an initial indication that the pressure approximation may exhibit enhanced convergence for this problem.

The corresponding convergence histories are shown in Figure 5.4 together with reference slopes representing the relevant experimental convergence rates.

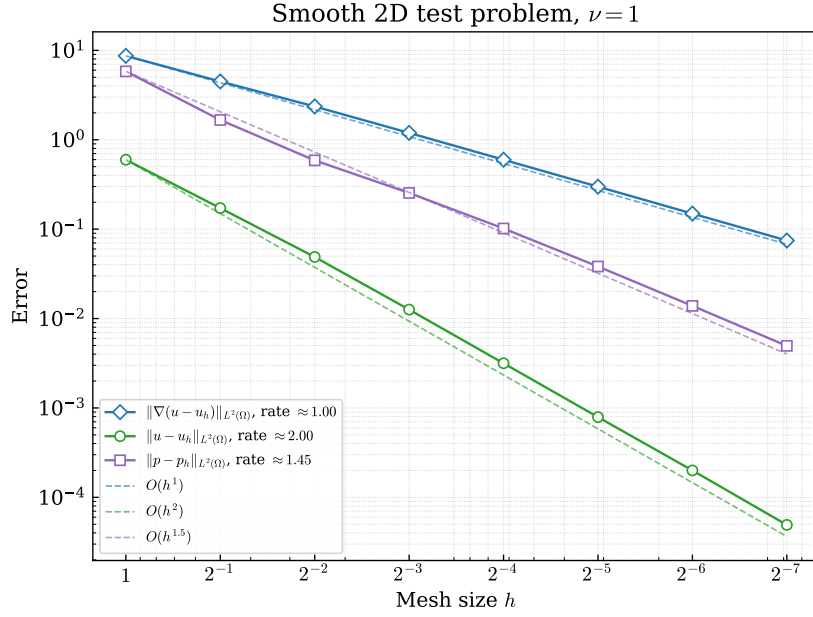


Figure 5.4: Convergence histories of the velocity and pressure errors for the smooth two-dimensional test problem with $\nu = 1$.

Numerical Results for $\nu = 10^{-6}$

We now repeat the same experiment with the viscosity reduced to $\nu = 10^{-6}$. The manufactured velocity and pressure fields and the complete mesh hierarchy are kept unchanged. Consequently, differences between Tables 5.2 and 5.3 can be attributed directly to the change in viscosity. This comparison is used to distinguish two effects: a change in the magnitude of the discretization errors and a change in their experimental convergence rates.

Table 5.3 shows that reducing the viscosity has a pronounced effect on the magnitude of the velocity errors. Compared with the results for $\nu = 1$, both velocity errors are substantially larger on corresponding refinement levels.

Table 5.3: Discretization errors for the smooth two-dimensional test problem with $\nu = 10^{-6}$.

Level	$\ \nabla(u - u_h)\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ p - p_h\ _{L^2(\Omega)}$	Rate
1	183424.56263	—	7171.82201	—	0.32736	—
2	50435.01200	1.86	951.64393	2.91	0.09329	1.81
3	12967.10985	1.96	98.13157	3.28	0.02060	2.18
4	3342.63687	1.96	11.52709	3.09	0.00501	2.04
5	843.29410	1.99	1.40107	3.04	0.00124	2.02
6	211.23914	2.00	0.17316	3.02	0.000307	2.01
7	52.81764	2.00	0.02155	3.01	7.65580e-05	2.00
8	13.20237	2.00	0.002689	3.00	1.91203e-05	2.00

At the same time, the experimental convergence rates exhibit a different higher order regime for this manufactured solution. On the finer meshes, the $H^1(\Omega)$ -seminorm velocity error approaches second order convergence, the $L^2(\Omega)$ -velocity error approaches

third order convergence, and the pressure error approaches second order convergence. These observed rates are specific to the present manufactured solution and viscosity regime and should not be interpreted as general convergence orders of the MINI finite element method. The corresponding convergence histories are presented in Figure 5.5.

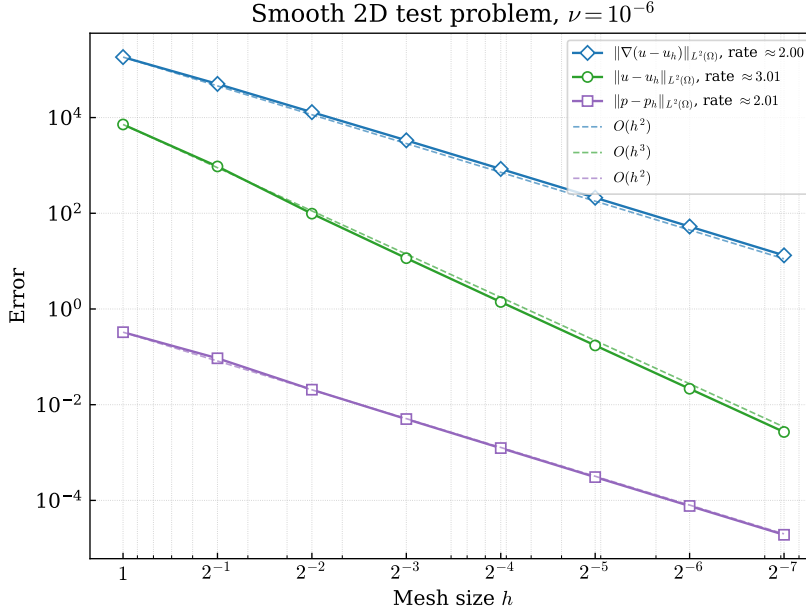


Figure 5.5: Convergence histories of the velocity and pressure errors for the smooth two-dimensional test problem with $\nu = 10^{-6}$.

The comparison of the two viscosity values, therefore, reveals two distinct effects. First, reducing ν strongly increases the magnitude of the velocity discretization errors. Second, the experimental convergence rates observed for the present manufactured solution also change, with higher order behaviour appearing for $\nu = 10^{-6}$. The pressure error, in contrast, remains much smaller in magnitude and exhibits enhanced convergence for both viscosity values. These observations motivate the following experiments, where the role of the manufactured pressure solution is examined more directly.

5.2 Two-Dimensional Test Problem with Non-Smooth Solution

We now consider a second manufactured solution taken from Pacheco and Steinbach [3]. In contrast to the smooth test problem considered in Section 5.1, the velocity solution in this example has reduced regularity, with $u \in H^2(\Omega) \setminus H^3(\Omega)$. In [3], this problem is used to illustrate the problem dependent nature of the pressure convergence for the MINI finite element.

The purpose of this experiment is to investigate whether the pressure convergence rate is lower than that observed for the smooth test problem in Section 5.1, where a rate approaching 1.5 was obtained for $\nu = 1$. In particular, the numerical results will

be compared with those reported for the MINI element in Table 3 of [3], where the pressure convergence remains below the 1.5 rate and approaches approximately first order behaviour on sufficiently refined meshes.

Following Pacheco and Steinbach [3], we consider the manufactured solution

$$u = \chi(x) \begin{pmatrix} (4x^5 - 6x^4 + 3x^3 - \frac{x^2}{2})\pi \sin(2\pi y) \\ x - 20x^4 + 24x^3 - 9x^2 \sin(\pi y) \end{pmatrix},$$

and

$$p = 3x^2 - 1,$$

where

$$\chi(x) := 1 + \operatorname{sign}\left(\frac{1}{2} - x\right).$$

The prescribed velocity and pressure fields are shown in Figures 5.6 and 5.7 respectively. The discontinuity introduced through $\chi(x)$ is responsible for the reduced regularity of the velocity field.

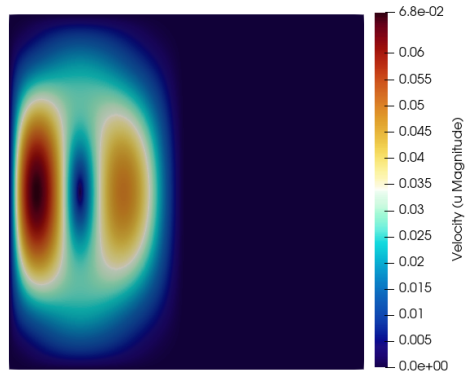


Figure 5.6: Velocity field for the non-smooth two-dimensional test problem visualized on a refined mesh.

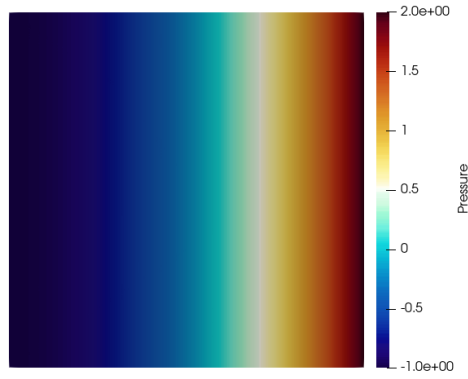


Figure 5.7: Pressure field for the non-smooth two-dimensional test problem visualised on a refined mesh.

Numerical Results for $\nu = 1$

The computations use the same initial mesh and uniform refinement strategy as in Section 5.1. The corresponding numbers of degrees of freedom are therefore those reported in Table 5.1.

Table 5.4 shows that the $H^1(\Omega)$ -seminorm velocity error approaches first order convergence, while the $L^2(\Omega)$ -velocity error exhibits approximately second order behaviour over most of the refinement sequence. Thus, the velocity approximation behaves consistently with the convergence pattern observed for $\nu = 1$ in the smooth test problem.

Table 5.4: Discretization errors for the non-smooth two-dimensional test problem with $\nu = 1$.

Level	$\ \nabla(u - u_h)\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ p - p_h\ _{L^2(\Omega)}$	Rate
1	0.21953	—	0.01186	—	0.08664	—
2	0.12310	0.83	0.00407	1.54	0.04264	1.02
3	0.06534	0.91	0.00116	1.81	0.01481	1.53
4	0.03346	0.97	0.00031	1.90	0.00604	1.29
5	0.01687	0.99	8.00e-05	1.95	0.00232	1.38
6	0.00847	0.99	2.10e-05	1.93	0.00085	1.45
7	0.00424	1.00	5.36e-06	1.97	0.00031	1.46
8	0.00212	1.00	1.64e-06	1.71	0.00011	1.49

The main observation concerns the pressure error. Its experimental convergence rate increases from approximately first order on the first refinement levels to values close to 1.5 on the finer meshes. In particular, the rates on the last three refinement levels are 1.45, 1.46, and 1.49.

This behaviour differs from the results reported in Table 3 of Pacheco and Steinbach [3] for the same manufactured solution and the MINI finite element. In their computations, the 1.5 pressure convergence rate is not reached and the experimental rate approaches approximately first order behaviour under continued refinement. The principal difference between the two numerical setups is the sequence of meshes used for the refinement study. The present results therefore indicate that the observed pressure convergence behaviour is sensitive to the chosen mesh sequence.

Figure 5.8 shows the corresponding convergence histories.

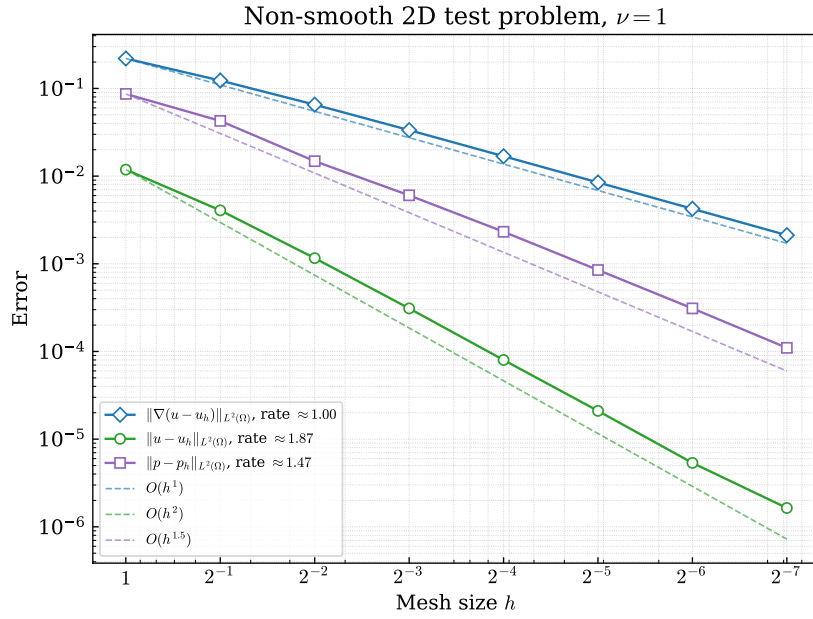


Figure 5.8: Convergence histories of the velocity and pressure errors for the non-smooth two-dimensional test problem with $\nu = 1$.

Numerical Results for $\nu = 10^{-6}$

To examine the influence of the viscosity parameter on this manufactured solution, we repeat the computation with $\nu = 10^{-6}$ while keeping the solution and mesh hierarchy unchanged. This additional experiment is considered separately from the comparison with Table 3 of [3], which corresponds to the Stokes problem with unit viscosity.

Since the manufactured solution is unchanged, the exact velocity and pressure fields are identical to those shown in Figures 5.6 and 5.7. They are therefore not repeated for this viscosity value.

The discretization errors are reported in Table 5.5, together with the experimental convergence rates.

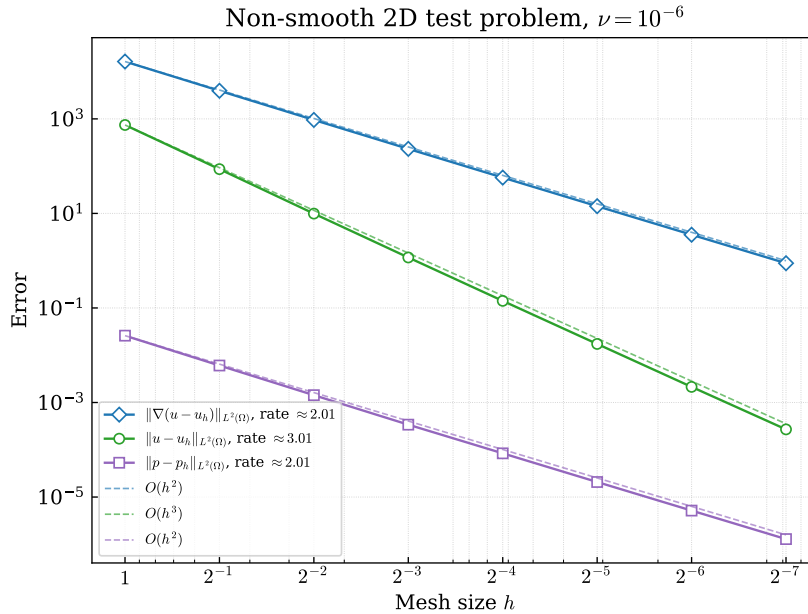
Table 5.5: Discretization errors for non-smooth two-dimensional test problem with $\nu = 10^{-6}$.

Level	$\ \nabla(u - u_h)\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ p - \tilde{p}^h\ _{L^2(\Omega)}$	Rate
1	16325.22705	—	742.83760	—	0.02595	—
2	3931.54434	2.05	86.46216	3.10	0.00606	2.10
3	949.57592	2.05	9.91233	3.12	0.00143	2.08
4	232.07417	2.03	1.16950	3.08	0.00034	2.07
5	57.26838	2.02	0.14125	3.05	8.41e-05	2.02
6	14.21785	2.01	0.01733	3.03	2.08e-05	2.02
7	3.54171	2.01	0.00214	3.02	5.17e-06	2.01
8	0.88381	2.00	0.00027	2.99	1.29e-06	2.05

The results in Table 5.5 exhibit a very regular higher order convergence regime. On the finer meshes, the experimental convergence rates are approximately two for the $H^1(\Omega)$ -seminorm velocity error, three for the $L^2(\Omega)$ -velocity error, and two for the $L^2(\Omega)$ -pressure error.

These rates differ substantially from those obtained for $\nu = 1$ in Table 5.4. In particular, the pressure convergence rate is approximately two rather than approaching 1.5. This shows that, for the present manufactured solution and mesh hierarchy, reducing the viscosity changes not only the magnitude of the velocity errors but also the experimentally observed convergence regime. These higher rates should therefore be regarded as specific to the present numerical setup rather than as general convergence orders of the MINI finite element.

The convergence histories for $\nu = 10^{-6}$ are shown in Figure 5.9.


 Figure 5.9: Convergence histories of the velocity and pressure errors for the non-smooth two-dimensional test problem with $\nu = 10^{-6}$.

The main purpose of this test problem was to investigate the lower pressure

convergence observed for the MINI element in the numerical example of Pacheco and Steinbach [3]. For $\nu = 1$, however, the present computations do not reproduce the approximately first order behaviour reported there. Instead, the pressure convergence rate increases towards 1.5 on the finer meshes. Since the same manufactured solution and finite element discretization are considered, this difference highlights the influence of the mesh sequence on the experimentally observed pressure convergence.

The additional computation with $\nu = 10^{-6}$ exhibits an even higher pressure convergence rate of approximately two. Thus, the numerical behaviour observed for this manufactured solution depends not only on the regularity of the solution but also on the particular discretization setting. The following experiment therefore considers a different pressure field in order to examine the role of the pressure structure and regularity more directly.

5.3 Two-Dimensional Test Problem with a Complicated Pressure Field

The previous examples showed enhanced pressure convergence for two different manufactured solutions. We now consider a problem designed to investigate how this behaviour is affected when the pressure field has a more pronounced localized structure. The pressure in the present example contains a sharp localized transition. As a consequence, its second order Sobolev seminorm is comparatively large. The purpose of this experiment is to determine whether this pressure structure suppresses or delays the onset of the enhanced pressure convergence observed in Sections 5.1 and 5.2.

We consider the Stokes problem on the unit square $\Omega = (0, 1)^2$, with homogeneous Dirichlet boundary conditions. The exact velocity is prescribed as $u = (0, 0)^T$ so that the problem represents a no-flow configuration. The pressure is defined by

$$p = 6x(1-x)y(1-y) \left(\frac{1}{2} + \frac{\arctan(2000(r_0^2 - (x-x_0)^2 - (y-y_0)))}{\pi} \right) - 0.172295197710, \quad (5.3.1)$$

where the constant is chosen such that $p \in L_0^2(\Omega)$ the pressure field is shown in Figure 5.10.

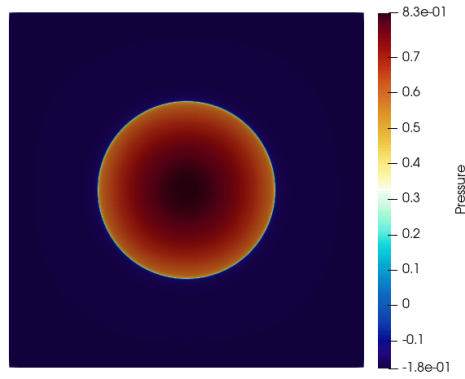


Figure 5.10: Approximation of the pressure field for two-dimensional test problem with a complicated pressure field.

The pressure field contains a sharp localized transition and therefore has a comparatively large second order Sobolev seminorm, $|p|_{H^2(\Omega)} \approx 3177.57$. This value is relevant to the refined pressure estimate because the terms that involve the $H^2(\Omega)$ -seminorm of the pressure can become dominant. Consequently, the mechanism leading to enhanced pressure convergence in the preceding examples is expected to be less pronounced on the considered mesh levels.

The same initial mesh and uniform refinement strategy as in Sections 5.1 and 5.2 are used. The corresponding numbers of velocity, pressure, and total degrees of freedom are therefore those reported in Table 5.1.

Numerical Results for $\nu = 1$

For $\nu = 1$, Table 5.6 shows a different pressure convergence pattern from those observed in Sections 5.1 and 5.2.

Table 5.6: Discretization errors for two-dimensional test problem with a complicated pressure field for $\nu = 1$.

Level	$ \nabla(u - u_h) _{L^2(\Omega)}$	Rate	$ u - u_h _{L^2(\Omega)}$	Rate	$ p - \tilde{p}^h _{L^2(\Omega)}$	Rate
1	0.04903	—	0.00317	—	0.24354	—
2	0.09263	-0.92	0.00412	-0.38	0.14511	0.75
3	0.05096	0.86	0.00228	0.85	0.09487	0.61
4	0.02676	0.93	0.00083	1.46	0.05170	0.88
5	0.01410	0.92	0.00021	1.98	0.02681	0.95
6	0.00705	1.00	4.17574e-05	2.33	0.01377	0.96
7	0.00318	1.15	5.40409e-06	2.95	0.00606	1.18
8	0.00112	1.51	6.07243e-07	3.15	0.00215	1.49

Over several intermediate refinement levels, the pressure error, converges at approximately first order, with experimental rates of 0.95 and 0.96 on levels 5 and

6. Thus, the higher order pressure convergence observed earlier is not visible for this pressure field. On the two finest levels, the pressure rate increases to approximately 1.18 and 1.49. The enhanced regime has therefore not disappeared completely. Instead, its onset is delayed to finer meshes. This behaviour is consistent with the large $H^2(\Omega)$ -seminorm of the pressure and supports the interpretation that the structure and regularity of the pressure field influence when enhanced convergence becomes visible.

The velocity errors also decrease with refinement. The $H^1(\Omega)$ -seminorm velocity error approaches approximately first order behaviour over the intermediate levels, while the $L^2(\Omega)$ -velocity error reaches higher experimental rates on the finest meshes. Since the exact velocity is zero, the main purpose of this experiment is the behaviour of the pressure approximation.

The corresponding convergence histories are shown in Figure 5.11.

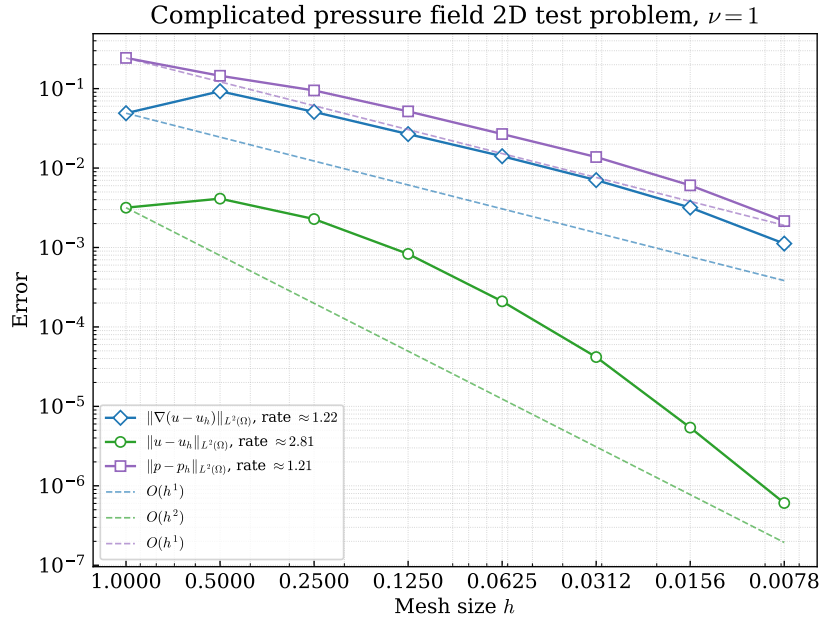


Figure 5.11: Convergence histories of the velocity and pressure errors for the complicated pressure field test problem with $\nu = 1$.

Numerical Results for $\nu = 10^{-6}$

Repeating the experiment with $\nu = 10^{-6}$ produces essentially the same pressure errors and experimental pressure convergence rates as for $\nu = 1$.

Table 5.7: Discretization errors for two-dimensional test problem with a complicated pressure field for $\nu = 10^{-6}$.

Level	$\ \nabla(u - u_h)\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ p - \tilde{p}^h\ _{L^2(\Omega)}$	Rate
1	49029.49318	—	3172.84760	—	0.243537	—
2	92633.32065	-0.92	4119.27204	-0.38	0.145107	0.75
3	50960.95890	0.86	2277.39858	0.86	0.094869	0.61
4	26756.26728	0.93	831.027908	1.45	0.051702	0.88
5	14103.28560	0.92	214.364475	1.95	0.026813	0.95
6	7048.238393	1.00	41.757410	2.36	0.013771	0.96
7	3179.974560	1.15	5.4040916	2.95	0.006058	1.18
8	1122.948173	1.51	0.6072433	3.15	0.002151	1.49

In contrast, the magnitude of the velocity errors changes substantially when the viscosity is reduced, although their experimental convergence pattern remains very similar. The comparison therefore separates the two effects clearly: the velocity error magnitude is strongly viscosity dependent, whereas the pressure convergence pattern in this example is governed primarily by the structure and regularity of the prescribed pressure field.

The pressure error is nearly identical for the two viscosity values. The pressure convergence rate is close to first order over the intermediate refinement levels, with rates of 0.95 and 0.96 on levels 5 and 6. On the finest levels, the rate increases to 1.18 and 1.49. This indicates that the pressure convergence behavior in this example is primarily related to the structure and regularity of the pressure field rather than to the viscosity.

The corresponding convergence histories are shown in Figure 5.12.

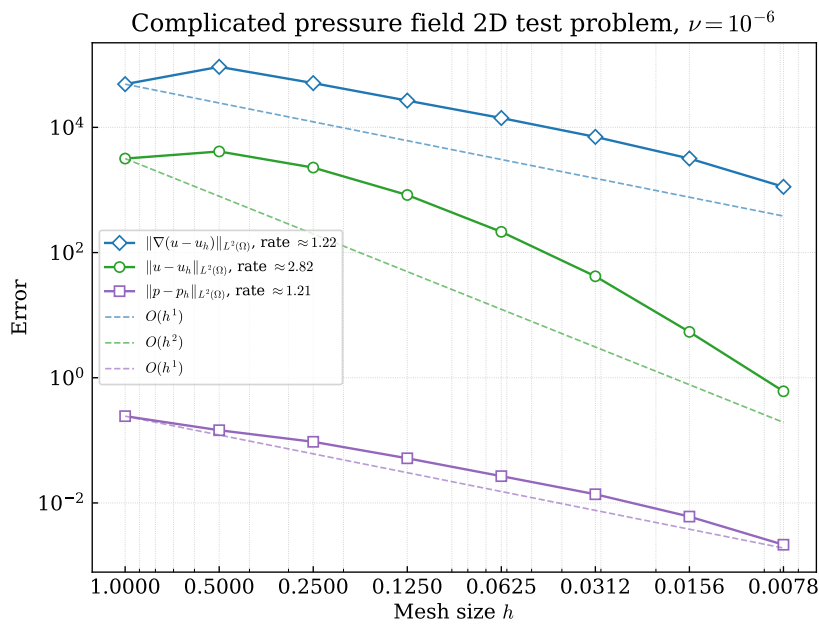


Figure 5.12: Convergence histories of the velocity and pressure errors for the complicated pressure field test problem with $\nu = 10^{-6}$.

Overall, this example demonstrates that the enhanced pressure convergence is sensitive to the structure and regularity of the pressure solution. Introducing a sharp localized transition and the associated large $H^2(\Omega)$ -seminorm substantially delays the enhanced convergence regime. For both viscosity values, the pressure error remains close to first order convergence over several intermediate refinement levels, while rates approaching those observed in the preceding examples become visible only on the finest meshes. The comparison with Sections 5.1 and 5.2 therefore indicates that enhanced pressure convergence is not a uniform feature across all manufactured pressure fields.

Summary of the two-dimensional numerical experiments.

In conclusion, the numerical experiments illustrate two distinct effects in the finite element approximation. The viscosity parameter has a pronounced influence on the magnitude of the velocity error, while the observed pressure convergence is strongly influenced by the structure and regularity of the manufactured pressure solution. For the first two test problems, enhanced pressure convergence is clearly visible on the refined meshes. For the pressure field containing a sharp localized transition, this enhanced regime is delayed and approximately first order convergence persists over several intermediate refinement levels. These observations are consistent with the role of the pressure regularity terms appearing in the refined estimates developed in Chapter 4.

5.4 Three-Dimensional Test Problem

The two-dimensional experiments in the previous sections confirm the convergence behaviour of the MINI element for the stationary Stokes problem. To examine whether the same behaviour carries over to three space dimensions, we now consider a three-dimensional manufactured solution on the unit cube. The test problem is based on the three-dimensional example introduced by Ganesan and John [4] in Section 4.2.

The original example of Ganesan and John was formulated for the stationary Navier–Stokes equations and was used to investigate pressure separation. Here, the manufactured velocity and pressure fields are instead used within the stationary Stokes equations (2.1.1). In particular, the convective term $(u \cdot \nabla)u$ is omitted, and the right-hand side $f = -\nu\Delta u + \nabla p$ is computed directly from the prescribed solution. Thus, the present experiment should be understood as an adaptation of the manufactured solution from [4], rather than as a reproduction of the numerical experiments reported there.

Exact Solution

We consider a three-dimensional test problem on the unit cube $\Omega = (0, 1)^3$. The velocity field $u = (u_1, u_2, u_3)^T$ is given by

$$\begin{aligned} u_1(x, y, z) &= \alpha \left(\sin(\pi x) \sin(\pi y) \sin(\pi z) + x^4 \cos(\pi y) \right), \\ u_2(x, y, z) &= \alpha \left(\cos(\pi x) \cos(\pi y) \cos(\pi z) - 3y^3 z \right), \\ u_3(x, y, z) &= \alpha \left(\cos(\pi x) \sin(\pi y) \cos(\pi z) + \cos(\pi x) \sin(\pi y) \sin(\pi z) \right. \\ &\quad \left. - 4x^3 z \cos(\pi y) + \frac{9}{2} y^2 z^2 \right). \end{aligned}$$

The pressure is prescribed by

$$p(x, y, z) = 3x - \sin(y + 4z) + c,$$

with the constant c is chosen such that $p \in L_0^2(\Omega)$. Numerically for this pressure field, $c \approx -1.239102$.

The parameter α corresponds to the parameter denoted by a in Ganesan and John [4]. To avoid a conflict with the bilinear form $a(\cdot, \cdot)$ used in this thesis, the parameter is denoted by α here. We choose $\alpha = 1$. A direct calculation shows that $\nabla \cdot u = 0$ so that the prescribed velocity field satisfies the incompressibility condition. The corresponding right-hand side is computed analytically from the stationary Stokes equations according to

$$f = -\nu \Delta u + \nabla p,$$

with $\nu = 1$. Consequently, (u, p) is the exact solution of the continuous Stokes problem.

Unlike the two-dimensional manufactured solutions considered previously, the velocity field in this example does not vanish on the whole boundary $\partial\Omega$. Therefore, the Dirichlet boundary data are taken directly from the exact solution, $g = u|_{\partial\Omega}$. The discrete problem is consequently solved with the corresponding inhomogeneous boundary values. This does not change the definition of the discretization errors considered below.

The prescribed fields are smooth on $\overline{\Omega}$. Hence, this example provides a three-dimensional counterpart to the smooth manufactured solution considered in Section 5.1, while using a different mesh hierarchy.

The domain is discretized by a uniform tetrahedral mesh. Each edge of the unit cube is divided into N equal parts, and each resulting sub-cube is subdivided into six tetrahedra. The computations are performed for

$$N \in \{2, 3, 4, 6, 8, 12, 16, 24, 32, 48\}.$$

For the present mesh sequence, the characteristic mesh size is taken as $h = \sqrt{3}/N$.

The MINI finite element is used on the tetrahedral meshes. For a mesh with parameter N , there are $(N + 1)^3$ vertices and $6N^3$ tetrahedral elements. The resulting numbers of velocity and pressure degrees of freedom are reported in Table 5.8.

Table 5.8: Degrees of freedom for the three-dimensional test problem.

N	h	Velocity DOFs	Pressure DOFs	Total DOFs
2	0.8660	225	27	252
3	0.5774	678	64	742
4	0.4330	1527	125	1652
6	0.2887	4917	343	5260
8	0.2165	11403	729	12132
12	0.1443	37695	2197	39892
16	0.1083	88467	4913	93380
24	0.0722	295707	15625	311332
32	0.0541	697635	35937	733572
48	0.0361	2343603	117649	2461252

Figure 5.13 shows the tetrahedral mesh for $N = 4$, corresponding to four subdivisions of each edge of the unit cube. A visualization of the three-dimensional solution itself is omitted, since the spatial structure of the velocity and pressure fields is difficult to represent clearly.

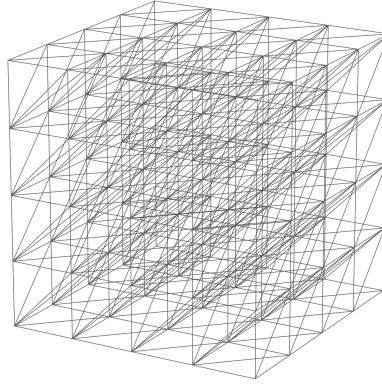


Figure 5.13: Grid for $N = 4$ for three-dimensional test problem.

The discretization errors are measured using the same norms as in the two-dimensional experiments,

$$\|\nabla(u - u_h)\|_{L^2(\Omega)}, \quad \|u - u_h\|_{L^2(\Omega)}, \quad \|p - p_h\|_{L^2(\Omega)}.$$

All three discretization errors decrease systematically as the mesh is refined. The $H^1(\Omega)$ -seminorm velocity error approaches first order convergence. The experimental rate decreases from approximately 1.35 on the first refinement step to 1.02 on the finest mesh. Thus, on the finest levels, the observed behaviour is consistent with $\|\nabla(u - u_h)\|_{L^2(\Omega)} = O(h)$.

Table 5.9: Discretization error for the three-dimensional test problem with $\nu = 1$ and $\alpha = 1$.

h	$\ \nabla(u - u_h)\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ p - p_h\ _{L^2(\Omega)}$	Rate
0.8660	1.071e + 01	–	6.182e - 01	–	4.727e + 01	–
0.5774	6.200e + 00	1.35	3.224e - 01	1.61	2.368e + 01	1.70
0.4330	4.301e + 00	1.27	1.775e - 01	2.08	1.498e + 01	1.59
0.2887	2.584e + 00	1.26	7.808e - 02	2.02	7.225e + 00	1.80
0.2165	1.834e + 00	1.19	4.362e - 02	2.02	4.247e + 00	1.85
0.1443	1.161e + 00	1.13	1.927e - 02	2.01	2.005e + 00	1.85
0.1083	8.510e - 01	1.08	1.082e - 02	2.01	1.182e + 00	1.84
0.0722	5.560e - 01	1.05	4.799e - 03	2.00	5.663e - 01	1.81
0.0541	4.134e - 01	1.03	2.698e - 03	2.00	3.388e - 01	1.79
0.0361	2.735e - 01	1.02	1.198e - 03	2.00	1.665e - 01	1.75

The $L^2(\Omega)$ -velocity error exhibits second order convergence over almost the entire refinement sequence. From the third mesh onwards, the experimental rates remain very close to two, and the final two refinement steps both give a rate of approximately 2. The observed behaviour is therefore consistent with $\|(u - u_h)\|_{L^2(\Omega)} = O(h^2)$.

The pressure error also decreases systematically under mesh refinement. Its experimental convergence rate is higher than first order over the considered mesh sequence and reaches values close to 1.85 on the intermediate and fine meshes. On the finest refinement level, the rate decreases slightly to 1.75. Overall, the results indicate an enhanced pressure convergence rate of approximately 1.8 for this three-dimensional test problem.

The corresponding convergence histories are shown in Figure [5.14](#). The velocity errors exhibit the expected first and second order slopes in the H^1 -seminorm and L^2 -norm, respectively. The pressure error decreases more rapidly than the first order behaviour observed in a standard approximation estimate.

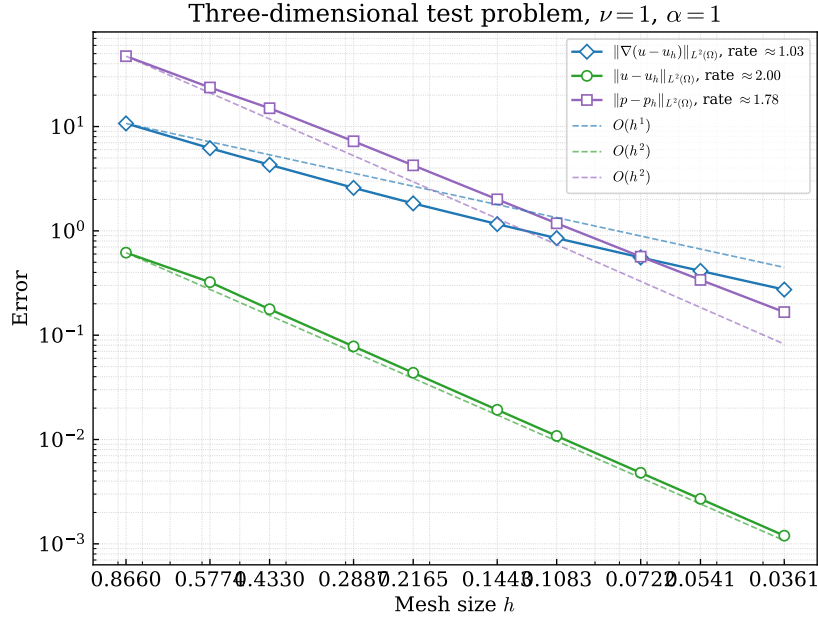


Figure 5.14: Convergence histories of the velocity and pressure errors for the smooth three-dimensional test problem with $\nu = 1$ and $\alpha = 1$.

Summary of the three-dimensional numerical experiment.

The three-dimensional results show the same qualitative convergence behaviour as the smooth two-dimensional test problem in Section 5.1. The $H^1(\Omega)$ -seminorm velocity error approaches first order convergence, while the $L^2(\Omega)$ -velocity error approaches second order convergence. In particular, the finest refinement levels give experimental rates of approximately 1.02 and 2, respectively.

The pressure error exhibits enhanced experimental convergence throughout most of the considered refinement sequence. The observed rate is approximately 1.7 - 1.85 on the intermediate and fine meshes and decreases slightly on the finest level. This behaviour is qualitatively similar to the enhanced pressure convergence observed in the preceding two-dimensional experiments. The pressure rate should nevertheless not be interpreted as a general asymptotic convergence order of the MINI element. As demonstrated by the two-dimensional experiments, the experimentally observed pressure behaviour depends on the manufactured solution, its regularity, and the mesh sequence. The present results provide numerical evidence that enhanced pressure convergence is also observed in three dimensions for this particular smooth manufactured solution.

Chapter 6

Conclusion

This thesis investigated the MINI finite element for the stationary incompressible Stokes problem from both theoretical and numerical perspectives. The analysis established the role of the bubble enrichment in obtaining a stable velocity-pressure discretization and examined the approximation properties and error behaviour of the resulting finite element pair.

The numerical experiments confirmed the expected convergence behaviour of the velocity approximation while revealing enhanced convergence behaviour for the pressure approximation. In particular, pressure convergence higher than the classical first order behaviour was observed for several manufactured solutions. The experiments demonstrated that this enhanced convergence is not characterized by a single universal rate: the observed behaviour depends on the pressure field, its regularity, the viscosity parameter, and the refinement regime. The example containing a sharp localized pressure transition was particularly illustrative, as the enhanced convergence became apparent only after sufficiently fine mesh refinement.

The three-dimensional experiment extends the numerical study beyond the two-dimensional setting. The results indicate that the expected first order H^1 -velocity and second order L^2 -velocity convergence persists for the MINI element in three dimensions. At the same time, the pressure approximation again exhibits a higher experimental convergence rate over the considered mesh sequence.

Overall, the results demonstrate that the MINI element provides the expected stable and convergent approximation for the stationary Stokes problem, while its pressure approximation may exhibit enhanced convergence that is strongly influenced by the particular problem under consideration. The numerical results therefore complement the theoretical estimates by illustrating the distinction between classical asymptotic error bounds and the convergence behaviour that can be observed in practical computations.

Bibliography

- [1] Volker John. *Finite element methods for incompressible flow problems*, volume 51 of *Springer Series in Computational Mathematics*. Springer, Cham, 2016.
- [2] V. Girault and P.-A. Raviart. *Finite element approximation of the Navier-Stokes equations*, volume 749 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1979.
- [3] Douglas R. Q. Pacheco and Olaf Steinbach. On the initial higher-order pressure convergence in equal-order finite element discretizations of the Stokes system. *Comput. Math. Appl.*, 109:140–145, 2022.
- [4] Sashikumaar Ganesan and Volker John. Pressure separation—a technique for improving the velocity error in finite element discretisations of the Navier-Stokes equations. *Appl. Math. Comput.*, 165(2):275–290, 2005.
- [5] Franco Brezzi. On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 8(R-2):129–151, 1974.
- [6] Douglas N. Arnold, Franco Brezzi, and Michel Fortin. A stable finite element for the Stokes equations. *Calcolo*, 21(4):337–344, 1984.
- [7] Michel Fortin. An analysis of the convergence of mixed finite element methods. *RAIRO Anal. Numér.*, 11(R-3):341–354, 1977.
- [8] Philippe Clément. Approximation by finite element functions using local regularization. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.*, 9(R-2):77–84, 1975.

Fachbereich Mathematik, Informatik und Physik

SELBSTSTÄNDIGKEITSERKLÄRUNG

Name: Kshetry	(BITTE nur Block- oder Maschinenschrift verwenden.)
Vorname(n): Gauri Singh	
Studiengang: Mathematik, M.Sc	
Matr. Nr.: 5577671	

Ich erkläre gegenüber der Freien Universität Berlin, dass ich die vorliegende Masterarbeit selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe.

Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Schriften entnommen sind, habe ich als solche kenntlich gemacht.

Diese Arbeit wurde in gleicher oder ähnlicher Form noch bei keiner anderen Universität als Prüfungsleistung eingereicht.

Datum: 25.09.2026

Unterschrift: 