Scientific Computing WS 2018/2019

Lecture 21

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Inhomogeneous Dirichlet problem: strong formulation

$$-\nabla \cdot \lambda \nabla u = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

- What can we say about minimum and maximum of the solution ?
- u has local local extremum in $x_0 \in \Omega$ if
  - x_0 is a critical point: $\nabla_u|_{x_0} = 0$
  - The matrix of second derivatives in x_0 is definite
  - This is linked to the sign of the right hand side: if $f = 0$ the main diagonal entries have different signs (as their sum is zero), so perhaps we would get a saddle point

Inhomogeneous Dirichlet problem: weak formulation

- Search $u \in H^1(\Omega)$ such that

$$u = u_g + \phi$$
$$\int_\Omega \lambda \nabla \phi \nabla v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x} - \int_\Omega \lambda \nabla u_g \nabla v \quad \forall v \in H_0^1(\Omega)$$

  Here, necessarily, $\phi \in H_0^1(\Omega)$ and we can apply the theory for the homogeneous Dirichlet problem.
- if u is a solution, we also have

$$\int_\Omega \lambda \nabla u \nabla v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega)$$

  as we can add $\int_\Omega \lambda \nabla u_g \nabla v$ on left and right side

Inhomogeneous Dirichlet problem: minimum principle

- Let $f \geq 0$.
- Let $g^\flat = \inf_{\partial\Omega} g$.
- Let $w = (u - g^\flat)^- = \min\{u - g^\flat, 0\} \in H_0^1(\Omega)$
- Consequently, $w \leq 0$
- As $\nabla u = \nabla(u - g^\flat)$ and $\nabla w = 0$ where $w \neq u - g^\flat$, one has

$$\begin{aligned} 0 \geq \int_\Omega fw \, d\mathbf{x} &= \int_\Omega \lambda \nabla u \nabla w \, d\mathbf{x} \\ &= \int_\Omega \lambda \nabla w \nabla w \, d\mathbf{x} \geq 0 \end{aligned}$$

- Therefore: $(u - g^\flat)^- = 0$ and $u \geq g^\flat$

Inhomogeneous Dirichlet problem: maximum principle

- Let $f \leq 0$.
- Let $g^\sharp = \sup_{\partial\Omega} g$.
- Let $w = (u - g^\sharp)^+ = \max\{u - g^\sharp, 0\} \in H_0^1(\Omega)$
- Consequently, $w \geq 0$
- As $\nabla u = \nabla(u - g^\sharp)$ and $\nabla w = 0$ where $w \neq u - g^\sharp$, one has

$$\begin{aligned} 0 \geq \int_\Omega fw\, d\mathbf{x} &= \int_\Omega \lambda \nabla u \nabla w\, d\mathbf{x} \\ &= \int_\Omega \lambda \nabla w \nabla w\, d\mathbf{x} \geq 0 \end{aligned}$$

- Therefore: $(u - g^\sharp)^- = 0$ and $u \leq g^\sharp$

Inhomogeneous Dirichlet problem: minmax principle

Theorem: The weak solution of the inhomogeneous Dirichlet problem

$$\begin{aligned} -\nabla \cdot \lambda \nabla u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

fulfills the global minimax principle: it attains its maximum at the boundary if $f \leq 0$ and attains its minimum at the boundary if $f \geq 0$.

Corollary: If $f = 0$ then u attains both its minimum and its maximum at the boundary.

Corolloary: Local minimax principle: This is true of any subdomain $\omega \subset \Omega$.

Convection-Diffusion problem

Green's theorem: If $w = 0$ on $\partial\Omega$:

$$\int_\Omega \mathbf{v} \cdot \nabla w \, d\mathbf{x} = -\int_\Omega w \nabla \cdot \mathbf{v} \, d\mathbf{x}$$

Let $\nabla \cdot \mathbf{v} = 0$. Search function $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla \cdot (D\nabla u - u\mathbf{v}) &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

From weak formulation (with Dirichlet lifting trick):

$$\int_\Omega (D\nabla u - u\mathbf{v}) \cdot \nabla w \, d\mathbf{x} = \int_\Omega f w \, d\mathbf{x} \quad \forall w \in H_0^1(\Omega)$$

Coercivity of bilinear form

Regard the convection contribution to the coercivity estimate:

$$
\begin{aligned}
-\int_\Omega u\mathbf{v}\cdot\nabla u\,d\mathbf{x} &= \int u\nabla\cdot(u\mathbf{v})\,d\mathbf{x} \quad \text{Green's theorem}\\
\int_\Omega u^2\nabla\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega u\mathbf{v}\cdot\nabla u\,d\mathbf{x} &= \int u\nabla\cdot(u\mathbf{v})\,d\mathbf{x} \quad \text{Product rule}\\
\int_\Omega u^2\nabla\cdot\mathbf{v}\,d\mathbf{x} + 2\int_\Omega u\mathbf{v}\cdot\nabla u\,d\mathbf{x} &= 0 \quad \text{Equation difference}\\
\int_\Omega u\mathbf{v}\cdot\nabla u\,d\mathbf{x} &= 0 \quad \text{Divergence condition} \nabla\cdot\mathbf{v} = 0
\end{aligned}
$$

Then

$$
\int_\Omega (D\nabla u - u\mathbf{v})\cdot\nabla u\,d\mathbf{x} = \int_\Omega D\nabla u\cdot\nabla u\,d\mathbf{x} \ge C||u||_{H_0^1(\Omega)}
$$

One could allow for fixed sign of $\nabla\cdot\mathbf{v}$.

Convection diffusion problem: maximum principle

- Let $f \leq 0$, $\nabla \cdot \mathbf{v} = 0$
- Let $g^\sharp = \sup_{\partial\Omega} g$.
- Let $w = (u - g^\sharp)^+ = \max\{u - g^\sharp, 0\} \in H_0^1(\Omega)$
- Consequently, $w \geq 0$
- As $\nabla u = \nabla(u - g^\sharp)$ and $\nabla w = 0$ where $w \neq u - g^\sharp$, one has

$$
\begin{aligned}
0 \geq \int_\Omega fw \, d\mathbf{x} &= \int_\Omega D(\nabla u - u\mathbf{v})\nabla w \, d\mathbf{x} \\
&= \int_\Omega D(\nabla w - w\mathbf{v})\nabla w \, d\mathbf{x} - Dg^\sharp \int_\Omega \mathbf{v} \cdot \nabla w \, d\mathbf{x} \\
&= \int_\Omega D\nabla w \cdot \nabla w \, d\mathbf{x} + Dg^\sharp \int_\Omega w\nabla \cdot \mathbf{v} \, d\mathbf{x} \\
&\geq C||w||_{H_0^1(\Omega)}
\end{aligned}
$$

- Therefore: $w = (u - g^\sharp)^- = 0$ and $u \leq g^\sharp$
- Similar for minimum part

Mimimax for convection-diffusion

Theorem: If $\nabla \cdot \mathbf{v} = 0$, the weak solution of the inhomogeneous Dirichlet problem

$$\begin{aligned} -\nabla \cdot (D\nabla u - u\mathbf{v}) &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

fulfills the global minimax principle: it attains its maximum at the boundary if $f \leq 0$ and attains its minimum at the boundary if $f \geq 0$.

Corollary: If $f = 0$ then u attains both its minimum and its maximum at the boundary.

Corolloary: Local minimax principle: This is true of any subdomain $\omega \subset \Omega$.

Interpretation of minimax principle

- Positive right hand side $\Rightarrow$ "production" of heat, matter ...
- No local minimum in the interior of domain if matter is produced.
- Also, positivity/nonnegativity of solutions if boundary conditions are positive/nonnegative
- Negative right hand side $\Rightarrow$ "consumption" of heat, matter ...
- No local maximum in the interior of domain if matter is consumed.
- Basic physical principle !

Discrete minimax principle

- $Au = f$
- A: matrix from diffusion or convection- diffusion
- A irreducibly diagonally dominant, positive main diagonal entries, negative off diagonal entries

$$a_{ii}u_i = \sum_{j\neq i} -a_{ij}u_j + f_i$$

$$u_i = \sum_{j\neq i, a_{ij}\neq 0} -\frac{a_{ij}}{a_{ii}}u_j + f_i$$

- For interior points,, $a_{ii} = -\sum_{j\neq i} a_{ij}$
- Assume i is interior point. Assume $f_i \geq 0 \Rightarrow$

$$u_i \geq \min_{j\neq i, a_{ij}\neq 0} u_j \sum_{j\neq i, a_{ij}\neq 0} -\frac{a_{ij}}{a_{ii}} = \min_{j\neq i, a_{ij}\neq 0} u_j$$

- Assume i is interior point. Assume $f_i \leq 0 \Rightarrow$

$$u_i \leq \max_{j\neq i, a_{ij}\neq 0} u_j \sum_{j\neq i, a_{ij}\neq 0} -\frac{a_{ij}}{a_{ii}} = \max_{j\neq i, a_{ij}\neq 0} u_j$$

Discussion of discrete minimax principle I

- P1 finite elements, Voronoi finite volumes: matrix graph $\equiv$ triangulation of domain
- The set $\{j \neq i, a_{ij} \neq 0\}$ is exactly the set of neigbor nodes
- Solution in point x_i estimated by solution in neigborhood
- The estimate can be propagated to the boundary of the domain

Discussion of discrete minimax principle II

- Minimax principle + positivity/nonnegativity of solutions can be seen as an important qualitative property of the physical process
- Along with good approximation quality, its preservation in the discretization process may be necessary
- Guaranteed for irreducibly diagonally dominant matrices
- Nonnegativity for nonnegative right hand sides guaranteed by M-Property
- Finite volume method may be preferred as it can guarantee these properties for boundary conforming Delaunay grids.

Convection-diffusion and finite elements

Search function $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla(\cdot D\nabla u - u\mathbf{v}) = f \quad &\text{in } \Omega\\
u = u_D \quad &\text{on } \partial\Omega
\end{aligned}
$$

- Assume v is divergence-free, i.e. $\nabla \cdot v = 0$.
- Then the main part of the equation can be reformulated as

  $$-\nabla(\cdot D\nabla u) + v \cdot \nabla u = 0 \quad \text{in } \Omega$$

  yielding a weak formulation: find $u \in H^1(\Omega)$ such that $u - u_D \in H^1_0(\Omega)$ and $\forall w \in H^1_0(\Omega)$,

  $$\int_\Omega D\nabla u \cdot \nabla w \; dx + \int_\Omega \mathbf{v} \cdot \nabla u \; w \; dx = \int_\Omega fw \; dx$$

- Galerkin formulation: find $u_h \in V_h$ with bc. such that $\forall w_h \in V_h$

  $$\int_\Omega D\nabla u_h \cdot \nabla w_h \; dx + \int_\Omega \mathbf{v} \cdot \nabla u_h \; w_h \; dx = \int_\Omega fw_h \; dx$$

Convection-diffusion and finite elements II

- Galerkin ansatz has similar problems as central difference ansatz in the finite volume/finite difference case $\Rightarrow$ stabilization ?
- Most popular: streamline upwind Petrov-Galerkin

$$\int_\Omega D\nabla u_h \cdot \nabla w_h \; dx + \int_\Omega \mathbf{v} \cdot \nabla u_h \; w_h \; dx + S(u_h, w_h) = \int_\Omega f w_h \; dx$$

with

$$S(u_h, w_h) = \sum_K \int_K (-\nabla(\cdot D\nabla u_h - u_h \mathbf{v}) - f)\delta_K v \cdot w_h \; dx$$

where $\delta_K = \frac{h_K^v}{2|\mathbf{v}|}\xi(\frac{|\mathbf{v}|h_K^v}{D})$ with $\xi(\alpha) = coth(\alpha) - \frac{1}{\alpha}$ and h_K^v is the size of element K in the direction of $\mathbf{v}$.

Convection-diffusion and finite elements III

- Many methods to stabilize, *none* guarantees M-Property even on weakly acute meshes ! (V. John, P. Knobloch, Computer Methods in Applied Mechanics and Engineering, 2007)
- Comparison paper:

  M. Augustin, A. Caiazzo, A. Fiebach, J. Fuhrmann, V. John, A. Linke, and R. Umla, "An assessment of discretizations for convection-dominated convection-diffusion equations," Comp. Meth. Appl. Mech. Engrg., vol. 200, pp. 3395–3409, 2011:

  > • if it is necessary to compute solutions without spurious oscillations: use FVM, taking care on the construction of an appropriate grid might be essential for reducing the smearing of the layers,
  >
  > • if sharpness and position of layers are important and spurious oscillations can be tolerated: often the SUPG method is a good choice.

- Topic of ongoing research

Time dependent Robin boundary value problem

- Choose final time $T > 0$. Regard functions $(x, t) \to \mathbb{R}$.

$$\begin{aligned} \partial_t u - \nabla \cdot \kappa \nabla u &= f \quad \text{in } \Omega \times [0, T] \\ \kappa \nabla u \cdot \vec{n} + \alpha(u - g) &= 0 \quad \text{on } \partial\Omega \times [0, T] \\ u(x, 0) &= u_0(x) \quad \text{in}\Omega \end{aligned}$$

- This is an initial boundary value problem
- This problem has a weak formulation in the Sobolev space $L^2\left([0, T], H^1(\Omega)\right)$, which then allows for a Galerkin approximation in a corresponding subspace
- We will proceed in a simpler manner: first, perform a finite difference discretization in time, then perform a finite element (finite volume) discretization in space.
  - *Rothe method:* first discretize in time, then in space
  - *Method of lines:* first discretize in space, get a huge ODE system, then apply perfom discretization

Time discretization

- Choose time discretization points $0 = t^0 < t^1 \dots < t^N = T$
- let $\tau^n = t^n - t^{n-1}$
  For $i = 1 \dots N$, solve

$$\begin{aligned} \frac{u^n - u^{n-1}}{\tau^n} - \nabla \cdot \kappa \nabla u_\theta = f \quad & \text{in } \Omega \times [0, T] \\ \kappa \nabla u_\theta \cdot \vec{n} + \alpha(u^\theta - g) = 0 \quad & \text{on } \partial\Omega \times [0, T] \end{aligned}$$

  where $u^\theta = \theta u^n + (1-\theta)u^{n-1}$
- $\theta = 1$: backward (implicit) Euler method
  Solve PDE problem in each timestep. First order accuracy in time.
- $\theta = \frac{1}{2}$: Crank-Nicolson scheme
  Solve PDE problem in each timestep. Second order accuracy in time.
- $\theta = 0$: forward (explicit) Euler method
  First order accurate in time. This does not involve the solution of a PDE problem $\Rightarrow$ Cheap? What do we have to pay for this ?

Finite volumes for time dependent problem

Search function $u : \Omega \times [0, T] \to \mathbb{R}$ such that $u(x,0) = u_0(x)$ and

$$\begin{aligned} \partial_t u - \nabla \cdot \lambda \nabla u &= 0 \quad \text{in} \Omega \times [0, T] \\ \lambda \nabla u \cdot \mathbf{n} &= 0 \quad \text{on} \Gamma \times [0, T] \end{aligned}$$

- Given control volume ω_k, integrate equation over space-time control volume $\omega_k \times (t^{n-1}, t^n)$, divide by τ^n:

$$\begin{aligned} 0 &= \int\limits_{\omega_k} \left(\frac{1}{\tau^n}(u^n - u^{n-1}) - \nabla \cdot \lambda \nabla u^\theta \right) d\omega \\ &= \frac{1}{\tau}^n \int\limits_{\omega_k} (u^n - u^{n-1}) d\omega - \int\limits_{\partial\omega_k} \lambda \nabla u^\theta \cdot \mathbf{n}_k d\gamma \\ &= -\sum_{l \in \mathcal{N}_k} \int\limits_{\sigma_{kl}} \lambda \nabla u^\theta \cdot \mathbf{n}_{kl} d\gamma - \int\limits_{\gamma_k} \lambda \nabla u^\theta \cdot \mathbf{n} d\gamma - \frac{1}{\tau}^n \int\limits_{\omega_k} (u^n - u^{n-1}) d\omega \\ &\approx \underbrace{\frac{|\omega_k|}{\tau^n}(u_k^n - u_k^{n-1})}_{\to M} + \underbrace{\sum_{l \in \mathcal{N}_k} \frac{|\sigma_{kl}|}{h_{kl}}(u_k^\theta - u_l^\theta)}_{\to A} \end{aligned}$$

Matrix equation

$$
\begin{aligned}
\frac{1}{\tau^n}\left(Mu^n - Mu^{n-1}\right) + Au^\theta = 0 & \\
\frac{1}{\tau^n}Mu^n + \theta Au^n &= \frac{1}{\tau^n}Mu^{n-1} + (\theta-1)Au^{n-1} \\
u^n + \tau^n M^{-1}\theta Au^n &= u^{n-1} + \tau^n M^{-1}(\theta-1)Au^{n-1}
\end{aligned}
$$

$M = (m_{kl})$, $A = (a_{kl})$ with

$$
a_{kl} = \begin{cases} \sum_{l' \in \mathcal{N}_k} \kappa \frac{|\sigma_{kl'}|}{h_{kl'}} & l = k \\ -\kappa \frac{\sigma_{kl}}{h_{kl}}, & l \in \mathcal{N}_k \\ 0, & else \end{cases}
$$

$$
m_{kl} = \begin{cases} |\omega_k| & l = k \\ 0, & else \end{cases}
$$

A matrix norm estimate

Lemma: Assume A has positive main diagonal entries, nonpositive off-diagonal entries and row sum zero. Then, $||(I+A)^{-1}||_\infty \leq 1$

Proof: Assume that $||(I+A)^{-1}||_\infty > 1$. $I+A$ is a irreducible M-matrix, thus $(I+A)^{-1}$ has positive entries. Then for α_{ij} being the entries of $(I+A)^{-1}$,

$$\max_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} > 1.$$

Let k be a row where the maximum is reached. Let $e = (1 \dots 1)^T$. Then for $v = (I+A)^{-1}e$ we have that $v > 0$, $v_k > 1$ and $v_k \geq v_j$ for all $j \neq k$. The kth equation of $e = (I+A)v$ then looks like

$$\begin{aligned} 1 &= v_k + v_k \sum_{j \neq k} |a_{kj}| - \sum_{j \neq k} |a_{kj}| v_j \\ &\geq v_k + v_k \sum_{j \neq k} |a_{kj}| - \sum_{j \neq k} |a_{kj}| v_k \\ &= v_k \\ &> 1 \end{aligned}$$

This contradiction enforces $||(I+A)^{-1}||_\infty \leq 1$. $\square$

Stability estimate

$$u^n + \tau^n M^{-1}\theta A u^n = u^{n-1} + \tau^n M^{-1}(\theta - 1)Au^{n-1} =: B^n u^{n-1}$$
$$u^n = (I + \tau^n M^{-1}\theta A)^{-1} B^n u^{n-1}$$

From the lemma we have $||(I + \tau^n M^{-1}\theta A)^n||_\infty \leq 1$ and $||u^n||_\infty \leq ||B^n u^{n-1}||_\infty$.

For the entries b^n_{kl} of B^n, we have

$$b^n_{kl} = \begin{cases} 1 + \frac{\tau^n}{m_{kk}}(\theta - 1)a_{kk}, & k = l \\ \frac{\tau^n}{m_{kk}}(\theta - 1)a_{kl}, & else \end{cases}$$

In any case, $b_{kl} \geq 0$ for $k \neq l$. If $b_{kk} \geq 0$, one estimates

$$||B||_\infty = \max_{k=1}^{N} \sum_{l=1}^{N} b_{kl}.$$

But

$$\sum_{l=1}^{N} b_{kl} = 1 + (\theta - 1)\frac{\tau^n}{m_{kk}}\left(a_{kk} + \sum_{l \in \mathcal{N}_k} a_{kl}\right) = 1$$

$$||B||_\infty = 1.$$

Stability conditions

- For a shape regular triangulation in $\mathbb{R}^d$, we can assume that $m_{kk} = |\omega_k| \sim h^d$, and $a_{kl} = \frac{|\sigma_{kl}|}{h_{kl}} \sim \frac{h^{d-1}}{h} = h^{d-2}$, thus $\frac{a_{kk}}{m_{kk}} \leq \frac{1}{Ch^2}$
- $b_{kk} \geq 0$ gives

$$(1-\theta)\frac{\tau^n}{m_{kk}} a_{kk} \leq 1$$

- A sufficient condition is

$$C(1-\theta)\frac{\tau^n}{Ch^2} \leq 1$$
$$(1-\theta)\tau^n \leq Ch^2$$

- Method stability:
  - Implicit Euler: $\theta = 1 \Rightarrow$ unconditional stability !
  - Explicit Euler: $\theta = 0 \Rightarrow$ CFL condition $\tau \leq Ch^2$
  - Crank-Nicolson: $\theta = \frac{1}{2} \Rightarrow$ CFL condition $\tau \leq 2Ch^2$
    Tradeoff stability vs. accuracy.

Stability discussion

- $\tau \leq Ch^2$ CFL == "Courant-Friedrichs-Levy"
- Explicit (forward) Euler method can be applied on very fast systems (GPU), with small time step comes a high accuracy in time.
- Implicit Euler: unconditional stability – helpful when stability is of utmost importance, and accuracy in time is less important
- For hyperbolic systems (pure convection without diffusion), the CFL conditions is $\tau \leq Ch$, thus in this case explicit computations are ubiquitous
- Comparison for a fixed size of the time interval. Assume for implicit Euler, time accuracy is less important, and the number of time steps is independent of the size of the space discretization.

	1D	2D	3D
# unknowns	$N = O(h^{-1})$	$N = O(h^{-2})$	$N = O(h^{-3})$
# steps	$M = O(N^2)$	$M = O(N)$	$M = O(N^{2/3})$
complexity	$M = O(N^3)$	$M = O(N^2)$	$M = O(N^{5/3})$

Backward Euler: discrete maximum principle

$$
\begin{aligned}
\frac{1}{\tau^n} M u^n + A u^n &= \frac{1}{\tau} M u^{n-1} \\
\frac{1}{\tau^n} m_{kk} u_k^n + a_{kk} u_k^n &= \frac{1}{\tau^n} m_{kk} u_k^{n-1} + \sum_{k \neq l} (-a_{kl}) u_l^n \\
u_k^n &= \frac{1}{\frac{1}{\tau^n} m_{kk} + \sum_{l \neq k} (-a_{kl})} \Big(\frac{1}{\tau^n} m_{kk} u_k^{n-1} + \sum_{l \neq k} (-a_{kl}) u_l^n \Big) \\
&\leq \frac{\frac{1}{\tau^n} m_{kk} + \sum_{l \neq k} (-a_{kl})}{\frac{1}{\tau^n} m_{kk} + \sum_{l \neq k} (-a_{kl})} \max(\{u_k^{n-1}\} \cup \{u_l^n\}_{l \in \mathcal{N}_k}) \\
&\leq \max(\{u_k^{n-1}\} \cup \{u_l^n\}_{l \in \mathcal{N}_k})
\end{aligned}
$$

- Provided, the right hand side is zero, the solution in a given node is bounded by the value from the old timestep, and by the solution in the neigboring points.
- No new local maxima can appear during time evolution
- There is a continuous counterpart which can be derived from weak solution
- Sign pattern is crucial for the proof.

Backward Euler: Nonnegativity

$$
\begin{aligned}
u^n + \tau^n M^{-1} A u^n &= u^{n-1} \\
u^n &= (I + \tau^n M^{-1} A)^{-1} u^{n-1}
\end{aligned}
$$

- $(I + \tau^n M^{-1} A)$ is an M-Matrix
- If $u_0 > 0$, then $u^n > 0 \; \forall n > 0$

Mass conservation

- Equivalent of $\int_\Omega \nabla\cdot\kappa\nabla u d\mathbf{x} = \int_{\partial\Omega}\kappa\nabla u\cdot\mathbf{n}d\gamma = 0$:

$$\begin{aligned}\sum_{k=1}^{N}\left(a_{kk}u_k + \sum_{l\in\mathcal{N}_k} a_{kl}u_l\right) &= \sum_{k=1}^{N}\sum_{l=1,l\neq k}^{N} a_{kl}(u_l-u_k)\\ &= \sum_{k=1}^{N}\sum_{l=1,l<k}^{N}\left(a_{kl}(u_l-u_k)+a_{lk}(u_k-u_l)\right)\\ &= 0\end{aligned}$$

- $\Rightarrow$ Equivalent of $\int_\Omega u^n d\mathbf{x} = \int_\Omega u^{n-1}d\mathbf{x}$:
  - $\sum_{k=1}^N m_{kk}u_k^n = \sum_{k=1}^N m_{kk}u_k^{n-1}$

Weak formulation of time step problem

- Weak formulation: search $u \in H^1(\Omega)$ such that $\forall v \in H^1(\Omega)$

$$\frac{1}{\tau^n}\int_\Omega u^n v\,dx + \theta \int_\Omega \kappa \nabla u^n \nabla v\,dx = \\ \frac{1}{\tau^n}\int_\Omega u^{n-1} v\,dx + (1-\theta)\int_\Omega \kappa \nabla u^{n-1} \nabla v\,dx$$

- Matrix formulation

$$\frac{1}{\tau^n} M u^n + \theta A u^n = \frac{1}{\tau^n} M u^{n-1} + (1-\theta) A u^{n-1}$$

- M: mass matrix, A: stiffness matrix.
- With FEM, Mass matrix lumping important for getting the previous estimates

Examination dates

Tue Feb 26.
Wed Feb 27.
Wed Mar 14.
Thu Mar 15.
Tue Mar 26.
Wed Mar 27.

Time: 10:00-13:00 (6 slots per examination date)

Please inscribe yourself into the corresponding sheets. (See also the back sides).

Room: t.b.a. (MA, third floor)

Prof. Nabben answers all administrative questions.