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Lecture 16

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The Galerkin method I

- Weak formulations "live" in Hilbert spaces which essentially are infinite dimensional
- For computer representations we need finite dimensional approximations
- The Galerkin method and its modifications provide a general scheme for the derivation of finite dimensional appoximations
- Finite dimensional subspaces of Hilbert spaces are the spans of a set of basis functions, and are Hilbert spaces as well $\Rightarrow$ e.g. the Lax-Milgram lemma is valid there as well

The Galerkin method II

- Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive with coercivity constant α, and continuity constant γ.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v) \, \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \, \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

Céa's lemma

- What is the connection between u and u_h ?
- Let $v_h \in V_h$ be arbitrary. Then

$$\begin{aligned}
\alpha||u-u_h||^2 &\leq a(u-u_h, u-u_h) \quad \text{(Coercivity)}\\
&= a(u-u_h, u-v_h) + a(u-u_h, v_h-u_h)\\
&= a(u-u_h, u-v_h) \quad \text{(Galerkin Orthogonality)}\\
&\leq \gamma||u-u_h|| \cdot ||u-v_h|| \quad \text{(Boundedness)}
\end{aligned}$$

- As a result

$$||u-u_h|| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u-v_h||$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace V_h.

From the Galerkin method to the matrix equation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of V_h.
- Then, we have the representation $u_h = \sum_{j=1}^n u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

  it is actually sufficient to require

$$\begin{aligned} a(u_h, \phi_i) &= f(\phi_i) \; (i = 1 \dots n) \\ a\left(\sum_{j=1}^n u_j \phi_j, \phi_i\right) &= f(\phi_i) \; (i = 1 \dots n) \\ \sum_{j=1}^n a(\phi_j, \phi_i) u_j &= f(\phi_i) \; (i = 1 \dots n) \\ AU &= F \end{aligned}$$

  with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j)$, $F = (f_i), f_i = F(\phi_i)$, $U = (u_i)$.
- Matrix dimension is $n \times n$. Matrix sparsity ?

Obtaining a finite dimensional subspace

- Let $\Omega = (a, b) \subset \mathbb{R}^1$
- Let $a(u, v) = \int_\Omega \lambda(x) \nabla u \nabla v d\mathbf{x}$.
- Analysis I provides a finite dimensional subspace: the space of sin/cos functions up to a certain frequency $\Rightarrow$ *spectral method*
- Ansatz functions have global support $\Rightarrow$ full $n \times n$ matrix
- OTOH: rather fast convergence for smooth data
- Generalization to higher dimensions possible
- Big problem in irregular domains: we need the eigenfunction basis of some operator...
- Spectral methods are successful in cases where one has regular geometry structures and smooth/constant coefficients – e.g. "Spectral Einstein Code"

Definition of a Finite Element (Ciarlet)

Triplet $\{K, P, \Sigma\}$ where

- $K \subset \mathbb{R}^d$: compact, connected Lipschitz domain with non-empty interior
- P: finite dimensional vector space of functions $p : K \to \mathbb{R}$
- $\Sigma = \{\sigma_1 \dots \sigma_s\} \subset \mathcal{L}(P, \mathbb{R})$: set of linear forms defined on P called *local degrees of freedom* such that the mapping

$$\begin{aligned} \Lambda_\Sigma : P &\to \mathbb{R}^s \\ p &\mapsto (\sigma_1(p) \dots \sigma_s(p)) \end{aligned}$$

is bijective, i.e. Σ is a basis of $\mathcal{L}(P, \mathbb{R})$.

Local shape functions

- Due to bijectivity of Λ_Σ, for any finite element $\{K, P, \Sigma\}$, there exists a basis $\{\theta_1 \dots \theta_s\} \subset P$ such that

$$\sigma_i(\theta_j) = \delta_{ij} \quad (1 \leq i, j \leq s)$$

- Elements of such a basis are called *local shape functions*

Unisolvence

- Bijectivity of Λ_Σ is equivalent to the condition

$$\forall(\alpha_1 \dots \alpha_s) \in \mathbb{R}^s \; \exists! p \in P \text{ such that } \sigma_i(p) = \alpha_i \quad (1 \leq i \leq s)$$

  i.e. for any given tuple of values $a = (\alpha_1 \dots \alpha_s)$ there is a unique polynomial $p \in P$ such that $\Lambda_\Sigma(p) = a$.
- Equivalent to *unisolvence*:

$$\begin{cases} \dim P = |\Sigma| = s \\ \forall p \in P: \; \sigma_i(p) = 0 \, (i = 1 \dots s) \; \Rightarrow \; p = 0 \end{cases}$$

Lagrange finite elements

- A finite element $\{K, P, \Sigma\}$ is called *Lagrange* finite element (or *nodal* finite element) if there exist a set of points $\{a_1 \dots a_s\} \subset K$ such that

$$\sigma_i(p) = p(a_i) \quad 1 \leq i \leq s$$

- $\{a_1 \dots a_s\}$: *nodes* of the finite element
- nodal basis: $\{\theta_1 \dots \theta_s\} \subset P$ such that

$$\theta_j(a_i) = \delta_{ij} \quad (1 \leq i, j \leq s)$$

Local interpolation operator

- Let $\{K, P, \Sigma\}$ be a finite element with shape function bases $\{\theta_1 \dots \theta_s\}$. Let $V(K)$ be a normed vector space of functions $v : K \to \mathbb{R}$ such that
  - $P \subset V(K)$
  - The linear forms in Σ can be extended to be defined on $V(K)$
- *local interpolation operator*

$$\begin{aligned} \mathcal{I}_K : V(K) &\to P \\ v &\mapsto \sum_{i=1}^{s} \sigma_i(v)\theta_i \end{aligned}$$

- P is invariant under the action of $\mathcal{I}_K$, i.e. $\forall p \in P, \mathcal{I}_K(p) = p$:
  - Let $p = \sum_{j=1}^{s} \alpha_j \theta_j$ Then,

$$\begin{aligned} \mathcal{I}_K(p) &= \sum_{i=1}^{s} \sigma_i(p)\theta_i = \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_j \sigma_i(\theta_j)\theta_i \\ &= \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_j \delta_{ij} \theta_i = \sum_{j=1}^{s} \alpha_j \theta_j \end{aligned}$$

Local Lagrange interpolation operator

- Let $V(K) = (\mathcal{C}^0(K))$

$$
\begin{aligned}
\mathcal{I}_K : V(K) &\to P \\
v &\mapsto I_K v = \sum_{i=1}^{s} v(a_i)\theta_i
\end{aligned}
$$

Simplices

- Let $\{a_0 \dots a_d\} \subset \mathbb{R}^d$ such that the d vectors $a_1 - a_0 \dots a_d - a_0$ are linearly independent. Then the convex hull K of $a_0 \dots a_d$ is called *simplex*, and $a_0 \dots a_d$ are called *vertices* of the simplex.
- *Unit simplex*: $a_0 = (0...0), a_1 = (0, 1 \dots 0) \dots a_d = (0 \dots 0, 1)$.

$$K = \left\{ x \in \mathbb{R}^d : x_i \geq 0 \; (i = 1 \dots d) \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}$$

- A general simplex can be defined as an image of the unit simplex under some affine transformation
- F_i: face of K opposite to a_i
- $\mathbf{n}_i$: outward normal to F_i

Barycentric coordinates

- Let K be a simplex.
- Functions λ_i ($i = 0 \dots d$):

$$\lambda_i : \mathbb{R}^d \to \mathbb{R}$$
$$x \mapsto \lambda_i(x) = 1 - \frac{(x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i}$$

where a_j is any vertex of K situated in F_i.

- For $x \in K$, one has

$$\begin{aligned} 1 - \frac{(x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} &= \frac{(a_j - a_i) \cdot \mathbf{n}_i - (x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} \\ &= \frac{(a_j - x) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} = \frac{\mathrm{dist}(x, F_i)}{\mathrm{dist}(a_i, F_i)} \\ &= \frac{\mathrm{dist}(x, F_i)|F_i|/d}{\mathrm{dist}(a_i, F_i)|F_i|/d} \\ &= \frac{\mathrm{dist}(x, F_i)|F_i|}{|K|} \end{aligned}$$

i.e. $\lambda_i(x)$ is the ratio of the volume of the simplex $K_i(x)$ made up of x and the vertices of F_i to the volume of K.

Barycentric coordinates II

- $\lambda_i(a_j) = \delta_{ij}$
- $\lambda_i(x) = 0 \; \forall x \in F_i$
- $\sum_{i=0}^{d} \lambda_i(x) = 1 \; \forall x \in \mathbb{R}^d$
  (just sum up the volumes)
- $\sum_{i=0}^{d} \lambda_i(x)(x - a_i) = 0 \; \forall x \in \mathbb{R}^d$
  (due to $\sum \lambda_i(x)x = x$ and $\sum \lambda_i a_i = x$ as the vector of linear coordinate functions)
- Unit simplex:
  - $\lambda_0(x) = 1 - \sum_{i=1}^{d} x_i$
  - $\lambda_i(x) = x_i$ for $1 \leq i \leq d$

Polynomial space $\mathbb{P}_k$

- Space of polynomials in $x_1 \dots x_d$ of total degree $\leq k$ with real coefficients $\alpha_{i_1 \dots i_d}$:

$$\mathbb{P}_k = \left\{ p(x) = \sum_{\substack{0 \leq i_1 \dots i_d \leq k \\ i_1 + \dots + i_d \leq k}} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} \right\}$$

- Dimension:

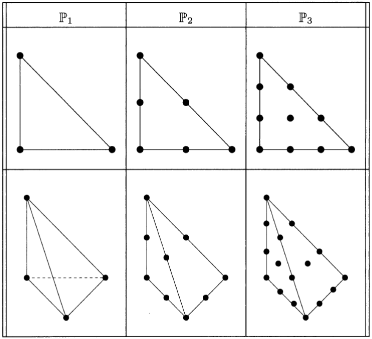
$$\dim \mathbb{P}_k = \binom{d+k}{k} = \begin{cases} k+1, & d = 1 \\ \frac{1}{2}(k+1)(k+2), & d = 2 \\ \frac{1}{6}(k+1)(k+2)(k+3), & d = 3 \end{cases}$$

$$\dim \mathbb{P}_1 = d + 1$$

$$\dim \mathbb{P}_2 = \begin{cases} 3, & d = 1 \\ 6, & d = 2 \\ 10, & d = 3 \end{cases}$$

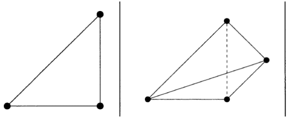
$\mathbb{P}_k$ simplex finite elements

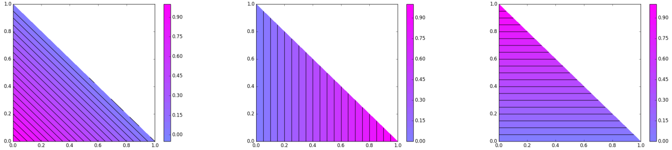
- K: simplex spanned by $a_0 \dots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_k$, such that $s = \dim P_k$
- For $0 \le i_0 \dots i_d \le k$, $i_0 + \dots + i_d = k$, let the set of nodes be defined by the points $a_{i_1 \dots i_d;k}$ with barycentric coordinates $(\frac{i_0}{k} \dots \frac{i_d}{k})$. Define Σ by $\sigma_{i_1 \dots i_d;k}(p) = p(a_{i_1 \dots i_d;k})$.



$\mathbb{P}_1$ simplex finite elements

- K: simplex spanned by $a_0 \ldots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_1$, such that $s = d + 1$
- Nodes $\equiv$ vertices
- Basis functions $\equiv$ barycentric coordinates

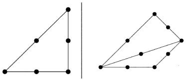


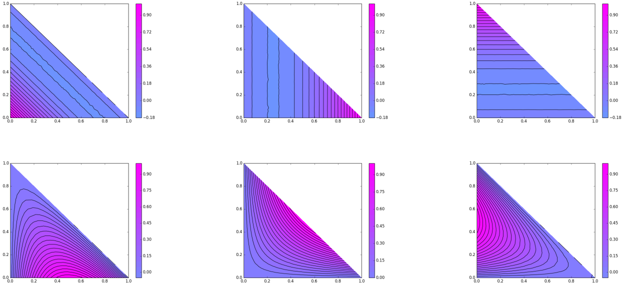


$\mathbb{P}_2$ simplex finite elements

- K: simplex spanned by $a_0 \dots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_2$, Nodes $\equiv$ vertices + edge midpoints
- Basis functions:

$$\lambda_i(2\lambda_i - 1), (0 \leq i \leq d); \quad 4\lambda_i\lambda_j, \quad (0 \leq i < j \leq d) \quad (\text{"edge bubbles"})$$





General finite elements

- Simplicial finite elements can be defined on triangulations of polygonal domains. During the course we will stick to this case.
- For vector PDEs, one can define finite elements for vector valued functions
- A curved domain Ω may be approximated by a polygonal domain Ω_h which is then triangulated. During the course, we will ignore this difference.
- As we have seen, more general elements are possible: cuboids, but also prismatic elements etc.
- Curved geometries are possible. Isoparametric finite elements use the polynomial space to define a mapping of some polyghedral reference element to an element with curved boundary

Conformal triangulations

- Let $\mathcal{T}_h$ be a subdivision of the polygonal domain $\Omega \subset \mathbb{R}^d$ into non-intersecting compact simplices K_m, $m = 1 \dots n_e$:

$$\overline{\Omega} = \bigcup_{m=1}^{n_e} K_m$$

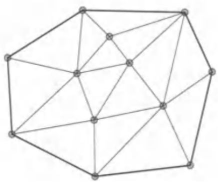
- Each simplex can be seen as the image of a affine transformation of a reference (e.g. unit) simplex $\widehat{K}$:

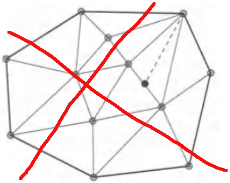
$$K_m = T_m(\widehat{K})$$

- We assume that it is conformal, i.e. if K_m, K_n have a $d-1$ dimensional intersection $F = K_m \cap K_n$, then there is a face $\widehat{F}$ of $\widehat{K}$ and renumberings of the vertices of K_n, K_m such that $F = T_m(\widehat{F}) = T_n(\widehat{F})$ and $T_m|_{\widehat{F}} = T_n|_{\widehat{F}}$

Conformal triangulations II

- $d = 1$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex
- $d = 2$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge





- $d = 3$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge or a common face
- Delaunay triangulations are conformal

Global interpolation operator $\mathcal{I}_h$

- Let $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ be a triangulation of Ω.
- Domain:

$$D(\mathcal{I}_h) = \{v \in (L^1(\Omega)) \text{ such that } \forall K \in \mathcal{T}_h, v|_K \in V(K)\}$$

- For all $v \in D(\mathcal{I}_h)$, define $\mathcal{I}_h v$ via

$$\mathcal{I}_h v|_K = \mathcal{I}_K(v|_K) = \sum_{i=1}^{s} \sigma_{K,i}(v|_K)\theta_{K,i} \; \forall K \in \mathcal{T}_h,$$

Assuming $\theta_{K,i} = 0$ outside of K, one can write

$$\mathcal{I}_h v = \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{s} \sigma_{K,i}(v|_K)\theta_{K,i},$$

mapping $D(\mathcal{I}_h)$ to the *approximation space*

$$W_h = \{v_h \in (L^1(\Omega)) \text{ such that } \forall K \in \mathcal{T}_h, v_h|_K \in P_K\}$$

H^1-Conformal approximation using Lagrangian finite elements

- Conformal subspace of W_h with zero jumps at element faces:

$$V_h = \{v_h \in W_h : \forall n, m, K_m \cap K_n \neq 0 \Rightarrow (v_h|_{K_m})_{K_m \cap K_n} = (v_h|_{K_n})_{K_m \cap K_n}\}$$

- Then: $V_h \subset H^1(\Omega)$.

Zero jump at interfaces with Lagrangian finite elements

- Assume geometrically conformal mesh
- Assume all faces of $\widehat{K}$ have the same number of nodes s^∂
- For any face $F = K_1 \cap K_2$ there are renumberings of the nodes of K_1 and K_2 such that for $i = 1 \dots s^\partial$, $a_{K_1,i} = a_{K_2,i}$
- Then, $v_h|_{K_1}$ and $v_h|_{K_2}$ match at the interface $K_1 \cap K_2$ if and only if they match at the common nodes

$$v_h|_{K_1}(a_{K_1,i}) = v_h|_{K_2}(a_{K_2,i}) \quad (i = 1 \dots s^\partial)$$

Global degrees of freedom

- Let $\{a_1 \dots a_N\} = \bigcup_{K \in \mathcal{T}_h} \{a_{K,1} \dots a_{K,s}\}$
- Degree of freedom map

$$
\begin{aligned}
j : \mathcal{T}_h \times \{1 \dots s\} &\to \{1 \dots N\} \\
(K, m) &\mapsto j(K, m) \text{ the global degree of freedom number}
\end{aligned}
$$

- Global shape functions $\phi_1, \dots, \phi_N \in W_h$ defined by

$$
\phi_i|_K(a_{K,m}) = \begin{cases} \delta_{mn} & \text{if } \exists n \in \{1 \dots s\} : j(K, n) = i \\ 0 & \text{otherwise} \end{cases}
$$

- Global degrees of freedom $\gamma_1, \dots, \gamma_N : V_h \to \mathbb{R}$ defined by

$$
\gamma_i(v_h) = v_h(a_i)
$$

Lagrange finite element basis

- $\{\phi_1, \dots, \phi_N\}$ is a basis of V_h, and $\gamma_1 \dots \gamma_N$ is a basis of $\mathcal{L}(V_h, \mathbb{R})$.

Proof:

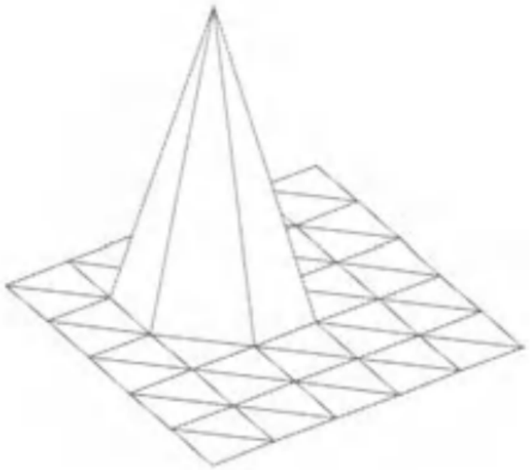
- $\{\phi_1, \dots, \phi_N\}$ are linearly independent: if $\sum_{j=1}^N \alpha_j \phi_j = 0$ then evaluation at $a_1 \dots a_N$ yields that $\alpha_1 \dots \alpha_N = 0$.
- Let $v_h \in V_h$. It is single valued in $a_1 \dots a_N$. Let $w_h = \sum_{j=1}^N v_h(a_j)\phi_j$. Then for all $K \in \mathcal{T}_h$, $v_h|_K$ and $w_h|_K$ coincide in the local nodes $a_{K,1} \dots a_{K,2}$, and by unisolvence, $v_h|_K = w_h|_K$.

Finite element approximation space

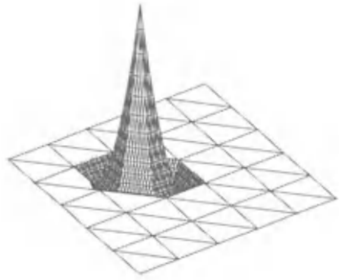
- $P^k_{c,h} = P^k_h = \{v_h \in \mathcal{C}^0(\bar{\Omega}_h) : \forall K \in \mathcal{T}_h, v_k \circ T_K \in \mathbb{P}^k\}$
- 'c' for continuity across mesh interfaces. There are also discontinuous FEM spaces which we do not consider here.

d	k	$N = \dim P^k_h$
1	1	N_v
1	2	$N_v + N_{el}$
1	3	$N_v + 2N_{el}$
2	1	N_v
2	2	$N_v + N_{ed}$
2	3	$N_v + 2N_{ed} + N_{el}$
3	1	N_v
3	2	$N_v + N_{ed}$
3	3	$N_v + 2N_{ed} + N_f$

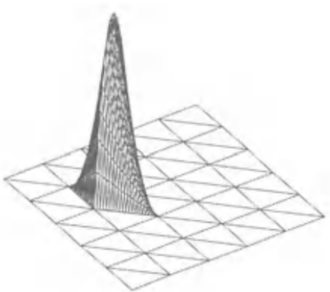
P^1 global shape functions



P^2 global shape functions



Node based



Edge based

Global Lagrange interpolation operator

Let $V_h = P_h^k$

$$\begin{aligned}\mathcal{I}_h : \mathcal{C}^0(\bar{\Omega}_h) &\to V_h\\ v &\mapsto \sum_{i=1}^{N} v(a_i)\phi_i\end{aligned}$$

Reference finite element

- Let $\{\widehat{P}, \widehat{K}, \widehat{\Sigma}\}$ be a fixed finite element
- Let T_K be some affine transformation and $K = T_K(\widehat{K})$
- There is a linear bijective mapping ψ_K between functions on K and functions on $\widehat{K}$:

$$\begin{aligned} \psi_K : V(K) &\to V(\widehat{K}) \\ f &\mapsto f \circ T_K \end{aligned}$$

- Let
  - $K = T_K(\widehat{K})$
  - $P_K = \{\psi_K^{-1}(\widehat{p}); \widehat{p} \in \widehat{P}\}$,
  - $\Sigma_K = \{\sigma_{K,i}, i = 1 \ldots s : \sigma_{K,i}(p) = \widehat{\sigma}_i(\psi_K(p))\}$

  Then $\{K, P_K, \Sigma_K\}$ is a finite element.
- This construction allows to develop theory for a reference element and to lift it later to an arbitrary element.

Commutativity of interpolation and reference mapping

- $\mathcal{I}_{\hat{K}} \circ \psi_K = \psi_K \circ \mathcal{I}_K$,
  i.e. the following diagram is commutative:

$$\begin{array}{ccc} V(K) & \xrightarrow{\psi_K} & V(\widehat{K}) \\ \downarrow{\scriptstyle \mathcal{I}_K} & & \downarrow{\scriptstyle \mathcal{I}_{\hat{K}}} \\ P_K & \xrightarrow{\psi_K} & P_{\widehat{K}} \end{array}$$

- $\equiv$ Interpolation and reference mapping are interchangeable

Affine transformation estimates I

- $\widehat{K}$: reference element
- Let $K \in \mathcal{T}_h$. Affine mapping described by matrix J_K and shift vertor b_K:

$$
\begin{aligned}
T_K : \widehat{K} &\to K \\
\widehat{x} &\mapsto J_K \widehat{x} + b_K
\end{aligned}
$$

  with $J_K \in \mathbb{R}^{d,d}, b_K \in \mathbb{R}^d$, J_K nonsingular

- Diameter of K: $h_K = \max_{x_1, x_2 \in K} ||x_1 - x_2||$
  $\equiv$ longest edge if K is triangular
- ρ_K diameter of largest ball that can be inscribed into K
- $\sigma_K = \frac{h_K}{\rho_K}$: local shape regularity measure
  $\sigma_K = 2\sqrt{3}$ for equilateral triangle
  $\sigma_K \to \infty$ if largest angle approaches π.

Affine transformation estimates II

Lemma T:

- $|\det J_K| = \frac{meas(K)}{meas(\widehat{K})}$
- $||J_K|| \leq \frac{h_K}{\rho_{\hat{K}}}$, $||J_K^{-1}|| \leq \frac{h_{\hat{K}}}{\rho_K}$
- $\Rightarrow ||J_K|| \cdot ||J_K^{-1}|| \leq c_{\hat{K}} \sigma_K$

Proof:

- $|\det J_K| = \frac{meas(K)}{meas(\widehat{K})}$: basic property of affine mappings
- Further:

$$||J_K|| = \sup_{\hat{x} \neq 0} \frac{||J_K \hat{x}||}{||\hat{x}||} = \frac{1}{\rho_{\hat{K}}} \sup_{||\hat{x}|| = \rho_{\hat{K}}} ||J_K \hat{x}||$$

  Set $\hat{x} = \hat{x}_1 - \hat{x}_2$ with $\hat{x}_1, \hat{x}_2 \in \widehat{K}$. Then $J_K \hat{x} = T_K \hat{x}_1 - T_K \hat{x}_2$ and one can estimate $||J_K \hat{x}|| \leq h_K$.

- For $||J_K^{-1}||$ regard the inverse mapping $\square$

Estimate of derivatives under affine transformation

- For $w \in H^s(K)$ recall the H^s seminorm $|w|^2_{s,K} = \sum_{|\beta|=s} ||\partial^\beta w||^2_{L^2(K)}$

Lemma D: Let $w \in H^s(K)$ and $\widehat{w} = w \circ T_K$. There exists a constant c such that

$$|\hat{w}|_{s,\hat{K}} \leq c||J_K||^s |\det J_K|^{-\frac{1}{2}} |w|_{s,K}$$
$$|w|_{s,K} \leq c||J_K^{-1}||^s |\det J_K|^{\frac{1}{2}} |\hat{w}|_{s,\hat{K}}$$

Proof: Let $|\alpha| = s$. By affinity and chain rule one obtains

$$||\partial^\alpha \hat{w}||_{L^2(\hat{K})} \leq c||J_K||^s \sum_{|\beta|=s} ||\partial^\beta w \circ T_K||_{L^2(K)}$$

Changing variables in the right hand side yields

$$||\partial^\alpha \hat{w}||_{L^2(\hat{K})} \leq c||J_K||^s |\det J_K|^{-\frac{1}{2}} |w|_{s,K}$$

Summation over α yields the first inequality. Regarding the inverse mapping yields the second estimate. □

Local interpolation error estimate I

Theorem: Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element with associated normed vector space $V(\widehat{K})$. Assume there exists k such that

$$\mathbb{P}_K \subset \widehat{P} \subset H^{k+1}(\widehat{K}) \subset V(\widehat{K})$$

and

$$H^{l+1}(\widehat{K}) \subset V(\widehat{K}) \quad \text{for} \quad 0 \leq l \leq k$$

Then there exists $c > 0$ such that for all $m = 0 \ldots l+1$, $K \in \mathcal{T}_h$, $v \in H^{l+1}(K)$:

$$|v - \mathcal{I}_K^k v|_{m,K} \leq c h_K^{l+1-m} \sigma_K^m |v|_{l+1,K}.$$

I.e. the the local interpolation error can be estimated through h_K, σ_K and the norm of a higher derivative.

Local interpolation error estimate II

Draft of Proof

- Estimate on reference element $\hat{K}$ using deeper results from functional analysis:

$$|\hat{w} - \mathcal{I}^k_{\hat{K}} \hat{w}|_{m,\hat{K}} \leq c|\hat{w}|_{l+1,\hat{K}} \qquad (*)$$

  (From Poincare like inequality, e.g. for $v \in H^1_0(\Omega)$, $c||v||_{L^2} \leq ||\nabla v||_{L^2}$: under certain circumstances, we can can estimate the norms of lower derivatives by those of the higher ones)

- Derive estimate on K from estimate on $\hat{K}$: Let $v \in H^{l+1}(K)$ and set $\hat{v} = v \circ T_K$. We know that $(\mathcal{I}^k_K v) \circ T_K = \mathcal{I}^k_{\hat{K}} \hat{v}$.

$$\begin{aligned}
|v - \mathcal{I}^k_K v|_{m,K} &\leq c||J_K^{-1}||^m |\det J_K|^{\frac{1}{2}} |\hat{v} - \mathcal{I}^k_{\hat{K}} \hat{v}|_{m,\hat{K}} && \text{(Lemma E)}\\
&\leq c||J_K^{-1}||^m |\det J_K|^{\frac{1}{2}} |\hat{v}|_{l+1,\hat{K}} && (*)\\
&\leq c||J_K^{-1}||^m ||J_K||^{l+1} |v|_{l+1,K} && \text{(Lemma E)}\\
&= c(||J_K|| \cdot ||J_K^{-1}||)^m ||J_K||^{l+1-m} |v|_{l+1,K} && \\
&\leq c h_K^{l+1-m} \sigma_K^m |v|_{l+1,K} && \text{(Lemma T)}
\end{aligned}$$

Local interpolation: special cases for Lagrange finite elements

- General condition

$$\mathbb{P}_K \subset \widehat{P} \subset H^{k+1}(\widehat{K}) \subset V(\widehat{K})$$
$$H^{l+1}(\widehat{K}) \subset V(\widehat{K}) \quad \text{for} \quad 0 \leq l \leq k$$

- $k = 1$:

$$\mathbb{P}_K \subset \widehat{P} \subset H^{2}(\widehat{K}) \subset V(\widehat{K})$$
$$H^{1}(\widehat{K}) \subset V(\widehat{K})$$

- $k = 1, l = 1, m = 0$: $|v - \mathcal{I}_K^k v|_{0,K} \leq c h_K^2 |v|_{2,K}$
- $k = 1, l = 1, m = 1$: $|v - \mathcal{I}_K^k v|_{1,K} \leq c h_K \sigma_K |v|_{2,K}$

Shape regularity

- Now we discuss a family of meshes $\mathcal{T}_h$ for $h \to 0$. We want to estimate global interpolation errors and see how they possibly diminuish
- For given $\mathcal{T}_h$, assume that $h = \max_{K \in \mathcal{T}_h} h_j$
- A family of meshes is called *shape regular* if

$$\forall h, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0$$

- In 1D, $\sigma_K = 1$
- In 2D, $\sigma_K \leq \frac{2}{\sin \theta_K}$ where θ_K is the smallest angle

Global interpolation error estimate

Theorem Let Ω be polyhedral, and let $\mathcal{T}_h$ be a shape regular family of affine meshes. Then there exists c such that for all h, $v \in H^{l+1}(\Omega)$,

$$||v - \mathcal{I}_h^k v||_{L^2(\Omega)} + \sum_{m=1}^{l+1} h^m \left(\sum_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^k v|^2_{m,K} \right)^{\frac{1}{2}} \leq c h^{l+1} |v|_{l+1,\Omega}$$

and

$$\lim_{h \to 0} \left(\inf_{v_h \in V_h^k} ||v - v_h||_{L^2(\Omega)} \right) = 0$$

Global interpolation error estimate for Lagrangian finite elements, $k = 1$

- Assume $v \in H^2(\Omega)$, e.g. if problem coefficients are smooth and the domain is convex

$$
\begin{aligned}
||v - \mathcal{I}_h^k v||_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} &\leq ch^2|v|_{2,\Omega} \\
|v - \mathcal{I}_h^k v|_{1,\Omega} &\leq ch|v|_{2,\Omega}
\end{aligned}
$$

$$
\lim_{h\to 0}\left(\inf_{v_h \in V_h^k} |v - v_h|_{1,\Omega}\right) = 0
$$

- If $v \in H^2(\Omega)$ cannot be guaranteed, estimates become worse. Example: L-shaped domain.
- These results immediately can be applied in Cea's lemma.

Error estimates for homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x} \; \forall v \in H_0^1(\Omega)$$

Then, $\lim_{h\to 0} ||u - u_h||_{1,\Omega} = 0$. If $u \in H^2(\Omega)$ (e.g. on convex domains) then

$$||u - u_h||_{1,\Omega} \leq ch|u|_{2,\Omega}$$
$$||u - u_h||_{0,\Omega} \leq ch^2|u|_{2,\Omega}$$

Under certain conditions (convex domain, smooth coefficients) one also has

$$||u - u_h||_{0,\Omega} \leq ch|u|_{1,\Omega}$$

("Aubin-Nitsche-Lemma")

H^2-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if Ω has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients (λ) are discontinuos
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simuations
  - Deterioration of convergence rate
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretizatiom mesh

Higher regularity

- If $u \in H^s(\Omega)$ for $s > 2$, convergence order estimates become even better for P^k finite elements of order $k > 1$.
- Depending on the regularity of the solution the combination of grid adaptation and higher oder ansatz functions may be successful