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Lecture 11

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Incomplete LU factorizations (ILU)

Idea (Varga, Buleev, 1960):

- fix a predefined zero pattern
- apply the standard LU factorization method, but calculate only those elements, which do not correspond to the given zero pattern
- Result: incomplete LU factors L, U, remainder R:

$$A = LU - R$$

- Problem: with complete LU factorization procedure, for any nonsingular matrix, the method is stable, i.e. zero pivots never occur. Is this true for the incomplete LU Factorization as well ?

Stability of ILU

Theorem (Saad, Th. 10.2): If A is an M-Matrix, then the algorithm to compute the incomplete LU factorization with a given nonzero pattern

$$A = LU - R$$

is stable. Moreover, $A = LU - R$ is a regular splitting.

ILU(0)

- Generally better convergence properties than Jacobi, Gauss-Seidel
- One can develop block variants
- Alternatives:
  - ILUM: ("modified"): add ignored off-diagonal entries to $\tilde{D}$
  - ILUT: zero pattern calculated dynamically based on drop tolerance
- Dependence on ordering
- Can be parallelized using graph coloring
- Not much theory: experiment for particular systems
- I recommend it as the default initial guess for a sensible preconditioner
- Incomplete Cholesky: symmetric variant of ILU

Meshes

- Regard boundary value problems for PDEs in a finite domain $\Omega \subset \mathbb{R}^d$
- Assume the domain is polygonal, its boundary $\partial\Omega$ is the union of a finite number of subsets of hyperplanes in $\mathbb{R}^n$ (line segments for $d = 2$, planar polygons for $d = 3$)
- A mesh (grid) is a subdivision Ω into a finite number of elementary closed (polygonal) subsets $T_1 \dots T_M$.
- Mostly, the elementary shapes are triangles or quadrilaterals ($d = 2$) or tetrahedra or cuboids ($d = 3$)
- During this course: focus on $d = 2$, triangles
- Synonymous: mesh = grid = triangulation

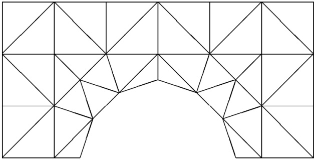
(FEM)-Admissible meshes

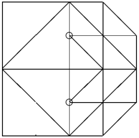
Definition: A grid is FEM-admissible if

(i) $\bar{\Omega} = \cup_{m=1}^{M} T_m$

(ii) If $T_m \cap T_n$ consists of exactly one point, then this point is a common vertex of T_m and T_n.

(iii) If for $m \neq n$, $T_m \cap T_n$ consists of more than one point, then $T_m \cap T_n$ is a common edge (or a common facet for $d = 3$) of T_m and T_n.





Source: Braess, FEM

Left: admissible mesh. Right: mesh with hanging nodes

Acute + weakly acute triangulations

Definition A triangulation of a domain Ω is

- acute, if all interior angles of all triangles are less than $\frac{\pi}{2}$,
- weakly acute, if all interior angles of all triangles are less than or equal to $\frac{\pi}{2}$.

Triangulation methods

- Geometrically most flexible
- Starting point for more general methods of subdivision into quadrilaterals
- Problem seems to be simple only at the first glance ...
- Here, we will discuss Delaunay triangulations, which have a number of interesting properties when it comes to PDE discretizations
  - J.R. Shewchuk: Lecture Notes on Delaunay Mesh Generation `http://web.mit.edu/ehliu/Public/ProjectX/Summer2005/delnotes.pdf`

Voronoi diagrams

After G. F. Voronoi, 1868-1908

Definition Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$. The set of points $H_{\mathbf{pq}} = \left\{ \mathbf{x} \in \mathbb{R}^d : ||\mathbf{x} - \mathbf{p}|| \leq ||\mathbf{x} - \mathbf{q}|| \right\}$ is the *half space* of points $\mathbf{x}$ closer to $\mathbf{p}$ than to $\mathbf{q}$.

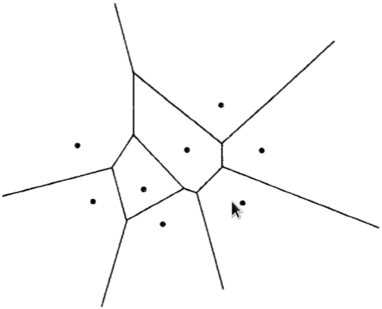
Definition Given a finite set of points $S \subset \mathbb{R}^d$, the *Voronoi region (Voronoi cell)* of a point $\mathbf{p} \in S$ is the set of points $\mathbf{x}$ closer to $\mathbf{p}$ than to any other point $\mathbf{q} \in S$:

$$V_{\mathbf{p}} = \left\{ \mathbf{x} \in \mathbb{R}^d : ||\mathbf{x} - \mathbf{p}|| \leq ||\mathbf{x} - \mathbf{q}|| \, \forall \mathbf{q} \in S \right\}$$

The *Voronoi diagram* of S is the collection of the Voronoi regions of the points of S.

Voronoi diagrams II

- The Voronoi diagram subdivides the whole space into “nearest neigbor” regions
- Being intersections of half planes, the Voronoi regions are convex sets



Voronoi diagram of 8 points in the plane

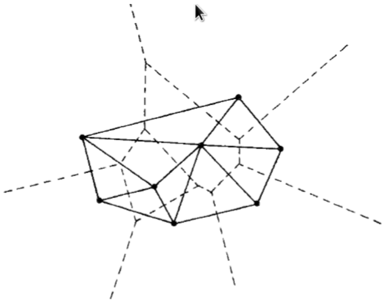
(H. Si)

Interactive example: http://homepages.loria.fr/BLevy/GEOGRAM/geogram_demo_Delaunay2d.html

Delaunay triangulation

After B.N. Delaunay (Delone), 1890-1980

- Assume that the points of S are in *general position*, i.e. no $d+2$ points of S are on one sphere (in 2D: no 4 points on one circle)
- Connect each pair of points whose Voronoi regions share a common edge with a line
- $\Rightarrow$ *Delaunay triangulation* of the convex hull of S



Delaunay triangulation of the convex hull of 8 points in the plane

(H. Si)

Delaunay triangulation II

- The circumsphere (circumcircle in 2D) of a d-dimensional simplex is the unique sphere containing all vertices of the simplex
- The circumball (circumdisc in 2D) of a simplex is the unique (open) ball which has the circumsphere of the simplex as boundary

Definition A triangulation of the convex hull of a point set S has the *Delaunay property* if each simplex (triangle) of the triangulation is Delaunay, i.e. its circumsphere (circumcircle) is empty wrt. S, i.e. it does not contain any points of S.

- The Delaunay triangulation of a point set S, where all points are in general position is unique
- Otherwise there is an ambiguity - if e.g. 4 points are one circle, there are two ways to connect them resulting in Delaunay triangles

Edge flips and locally Delaunay edges (2D only)

- For any two triangles **abc** and **adb** sharing a common edge **ab**, there is the *edge flip* operation which reconnects the points in such a way that two new triangles emerge: **adc** and **cdb**.
- An edge of a triangulation is locally Delaunay if it either belongs to exactly one triangle, or if it belongs to two triangles, and their respective circumdisks do not contain the points opposite wrt. the edge
- If an edge is locally Delaunay and belongs to two triangles, the sum of the angles opposite to this edge is less or equal to π.
- If all edges of a triangulation of the convex hull of S are locally Delaunay, then the triangulation is the Delaunay triangulation
- If an edge is not locally Delaunay and belongs to two triangles, the edge emerging from the corresponding edge flip will be locally Delaunay

Edge flip algorithm (Lawson)

Input: A stack L of edges of a given triangulation of S;

```
while L ≠ ∅ do
    pop an edge ab from L;
    if ab is not locally Delaunay then
        flip ab to cd;
        push edges ac, cb, db, da onto L;
    end
end
```

- This algorithm is known to terminate. After termination, all edges will be locally Delaunay, so the output is the Delaunay triangulation of S.
- Among all triangulations of a finite point set S, the Delaunay triangulation maximises the minimum angle
- All triangulations of S are connected via a flip graph

Radomized incremental flip algorithm (2D only)

- Create Delaunay triangulation of point set S by inserting points one after another, and creating the Delaunay triangulation of the emerging subset of S using the flip algorithm
- Estimated complexity: $O(n \log n)$
- In 3D, there is no simple flip algorithm, generalizations are active research subject

Triangulations of finite domains

- So far, we discussed triangulations of point sets, but in practice, we need triangulations of domains
- Create Delaunay triangulation of point set, "Intersect" with domain

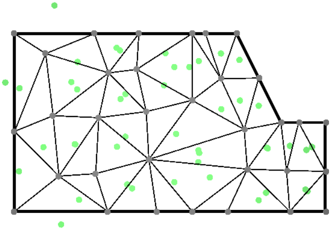
Boundary conforming Delaunay triangulations

Definition: An admissible triangulation of a polygonal Domain $\Omega \subset \mathbb{R}^d$ has the boundary conforming Delaunay property if

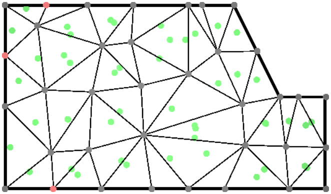
(i) All simplices are Delaunay

(ii) All boundary simplices (edges in 2D, facets in 3d) have the Gabriel property, i.e. their minimal circumdisks are empty

- Equivalent definition in 2D: sum of angles opposite to interior edges $\leq \pi$, angle opposite to boundary edge $\leq \frac{\pi}{2}$
- Creation of boundary conforming Delaunay triangulation description may involve insertion of Steiner points at the boundary



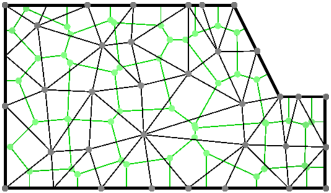
Delaunay grid of Ω



Boundary conforming Delaunay grid of Ω

Domain blendend Voronoi cells

- For Boundary conforming Delaunay triangulations, the intersection of the Voronoi diagram with the domain yields a well defined dual subdivision



Boundary conforming Delaunay triangulations II

- Weakly acute triangulations are boundary conforming Delaunay, but not vice versa!
- Working with weakly acute triangulations for general polygonal domains is unrealistic, especially in 3D
- For boundary conforming Delaunay triangulations of polygonal domains there are algoritms with mathematical termination proofs valid in many relevant cases
- Code examples:
  - 2D: Triangle by J.R.Shewchuk `https://www.cs.cmu.edu/~quake/triangle.html`
  - 3D: TetGen by H. Si `http://tetgen.org`
- Features:
  - polygonal geometry description
  - automatic insertion of points according to given mesh size criteria
  - accounting for interior boundaries
  - local mesh size control for a priori refinement
  - quality control
  - standalone executable & library

Further mesh generation approaches

(Most of them lose Delaunay property)

- Advancing front: create mesh of boundary, “grow” triangles from boundary to interior implemented e.g. in netgen by J. Schöberl https://sourceforge.net/projects/netgen-mesher/
- Quadtree/octree: place points on quadtree/octree hierachy and triangulate
- Mesh improvement: equilibrate element sizes + quality by iteratively modifying point locations
- ... active research topic with many open questions, unfortunately not exactly mainstream ...