Scientific Computing WS 2018/2019

Lecture 9

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Simple iteration with preconditioning

Idea: $A\hat{u} = b \Rightarrow$

$$\hat{u} = \hat{u} - M^{-1}(A\hat{u} - b)$$

$\Rightarrow$ iterative scheme

$$u_{k+1} = u_k - M^{-1}(Au_k - b) \quad (k = 0, 1 \ldots)$$

1. Choose initial value u_0, tolerance ε, set $k = 0$
2. Calculate *residuum* $r_k = Au_k - b$
3. Test convergence: if $||r_k|| < \varepsilon$ set $u = u_k$, finish
4. Calculate *update*: solve $Mv_k = r_k$
5. Update solution: $u_{k+1} = u_k - v_k$, set $k = i + 1$, repeat with step 2.

The Jacobi method

- Let $A = D - E - F$, where D: main diagonal, E: negative lower triangular part F: negative upper triangular part
- Preconditioner: $M = D$, where D is the main diagonal of $A \Rightarrow$

$$u_{k+1,i} = u_{k,i} - \frac{1}{a_{ii}}\left(\sum_{j=1\dots n} a_{ij}u_{k,j} - b_i\right) \quad (i = 1\dots n)$$

- Equivalent to the succesive (row by row) solution of

$$a_{ii}u_{k+1,i} + \sum_{j=1\dots n, j\neq i} a_{ij}u_{k,j} = b_i \quad (i = 1\dots n)$$

- Already calculated results not taken into account
- Alternative formulation with $A = M - N$:

$$\begin{aligned} u_{k+1} &= D^{-1}(E+F)u_k + D^{-1}b \\ &= M^{-1}Nu_k + M^{-1}b \end{aligned}$$

- Variable ordering does not matter

The Gauss-Seidel method

- Solve for main diagonal element row by row
- Take already calculated results into account

$$a_{ii}u_{k+1,i} + \sum_{j<i} a_{ij}u_{k+1,j} + \sum_{j>i} a_{ij}u_{k,j} = b_i \qquad (i = 1 \dots n)$$

$$(D - E)u_{k+1} - Fu_k = b$$

- May be it is faster
- Variable order probably matters
- Preconditioners: forward $M = D - E$, backward: $M = D - F$
- Splitting formulation: $A = M - N$
  forward: $N = F$, backward: $M = E$
- Forward case:

$$\begin{aligned} u_{k+1} &= (D - E)^{-1}Fu_k + (D - E)^{-1}b \\ &= M^{-1}Nu_k + M^{-1}b \end{aligned}$$

Block methods

- Jacobi, Gauss-Seidel, (S)SOR methods can as well be used block-wise, based on a partition of the system matrix into larger blocks,
- The blocks on the diagonal should be square matrices, and invertible
- Interesting variant for systems of partial differential equations, where multiple species interact with each other

Convergence

- Let $\hat{u}$ be the solution of $Au = b$.
- Let $e_k = u_j - \hat{u}$ be the error of the k-th iteration step

$$
\begin{aligned}
u_{k+1} &= u_k - M^{-1}(Au_k - b)\\
&= (I - M^{-1}A)u_k + M^{-1}b\\
u_{k+1} - \hat{u} &= u_k - \hat{u} - M^{-1}(Au_k - A\hat{u})\\
&= (I - M^{-1}A)(u_k - \hat{u})\\
&= (I - M^{-1}A)^k(u_0 - \hat{u})
\end{aligned}
$$

resulting in

$$
e_{k+1} = (I - M^{-1}A)^k e_0
$$

- So when does $(I - M^{-1}A)^k$ converge to zero for $k \to \infty$?

Spectral radius and convergence

Definition The spectral radius $\rho(A)$ is the largest absolute value of any eigenvalue of A: $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$.

Theorem (Saad, Th. 1.10) $\lim_{k\to\infty} A^k = 0 \Leftrightarrow \rho(A) < 1$.

Proof, $\Rightarrow$: Let u_i be a unit eigenvector associated with an eigenvalue λ_i. Then

$$
\begin{aligned}
Au_i &= \lambda_i u_i \\
A^2 u_i &= \lambda_i A_i u_i = \lambda^2 u_i \\
&\vdots \\
A^k u_i &= \lambda^k u_i \\
\text{therefore} \quad ||A^k u_i||_2 &= |\lambda^k| \\
\text{and} \quad \lim_{k\to\infty} |\lambda^k| &= 0
\end{aligned}
$$

so we must have $\rho(A) < 1$

Corollary from proof

Theorem (Saad, Th. 1.12)

$$\lim_{k\to\infty} ||A^k||^{\frac{1}{k}} = \rho(A)$$

□

Back to iterative methods

Sufficient condition for convergence: $\rho(I - M^{-1}A) < 1$.

Eigenvalue analysis for more general matrices

- For 1D heat conduction we used a very special regular structure of the matrix which allowed exact eigenvalue calculations
- Generalizations to tensor product is possible
- Generalization to varying coefficients, unstructured grids ...
  $\Rightarrow$ what can be done for general matrices ?

The Gershgorin Circle Theorem (Semyon Gershgorin,1931)

(everywhere, we assume $n \geq 2$)

Theorem (Varga, Th. 1.11) Let A be an $n \times n$ (real or complex) matrix. Let

$$\Lambda_i = \sum_{\substack{j=1\dots n \\ j\neq i}} |a_{ij}|$$

If λ is an eigenvalue of A then there exists r, $1 \leq r \leq n$ such that

$$|\lambda - a_{rr}| \leq \Lambda_r$$

Proof Assume λ is eigenvalue, $\mathbf{x}$ a corresponding eigenvector, normalized such that $\max_{i=1\dots n} |x_i| = |x_r| = 1$. From $A\mathbf{x} = \lambda\mathbf{x}$ it follows that

$$(\lambda - a_{ii})x_i = \sum_{\substack{j=1\dots n \\ j\neq i}} a_{ij}x_j$$

$$|\lambda - a_{rr}| = |\sum_{\substack{j=1\dots n \\ j\neq r}} a_{rj}x_j| \leq \sum_{\substack{j=1\dots n \\ j\neq r}} |a_{rj}||x_j| \leq \sum_{\substack{j=1\dots n \\ j\neq r}} |a_{rj}| = \Lambda_r$$

□

Gershgorin Circle Corollaries

Corollary: Any eigenvalue of A lies in the union of the disks defined by the Gershgorin circles

$$\lambda \in \bigcup_{i=1\dots n} \{\mu \in \mathbb{V} : |\mu - a_{ii}| \leq \Lambda_i\}$$

Corollary:

$$\rho(A) \leq \max_{i=1\dots n} \sum_{j=1}^{n} |a_{ij}| = ||A||_\infty$$

$$\rho(A) \leq \max_{j=1\dots n} \sum_{i=1}^{n} |a_{ij}| = ||A||_1$$

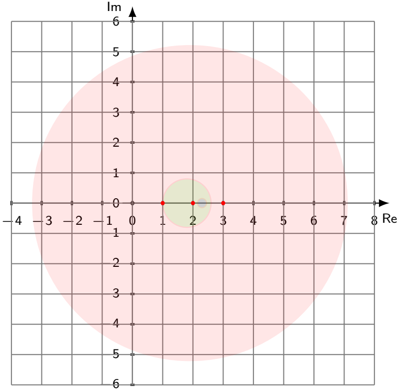
Proof

$$|\mu - a_{ii}| \leq \Lambda_i \quad \Rightarrow \quad |\mu| \leq \Lambda_i + |a_{ii}| = \sum_{j=1}^{n} |a_{ij}|$$

Furthermore, $\sigma(A) = \sigma(A^T)$. □

Gershgorin circles: example

$$A = \begin{pmatrix} 1.9 & 1.8 & 3.4 \\ 0.4 & 1.8 & 0.4 \\ 0.05 & 0.1 & 2.3 \end{pmatrix}, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \Lambda_1 = 5.2, \Lambda_2 = 0.8, \lambda_3 = 0.15$$



Gershgorin circles: heat example I

$$A = \begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{2}{h} \end{pmatrix}$$

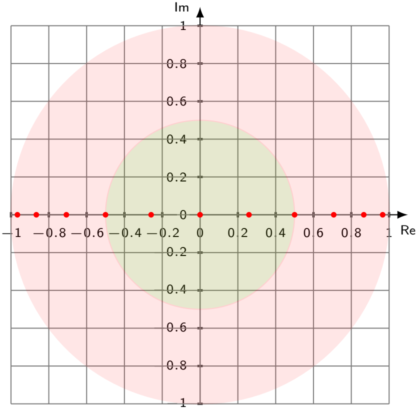
$$B = (I - D^{-1}A) = \begin{pmatrix} 0 & \frac{1}{2} & & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{pmatrix}$$

We have $b_{ii} = 0$, $\Lambda_i = \begin{cases} \frac{1}{2}, & i = 1, n \\ 1 & i = 2 \dots n-1 \end{cases}$ $\Rightarrow$ estimate $|\lambda_i| \le 1$

Gershgorin circles: heat example II

Let n=11, h=0.1:

$$\lambda_i = \cos\left(\frac{ih\pi}{1+2h}\right) \quad (i = 1 \dots n)$$



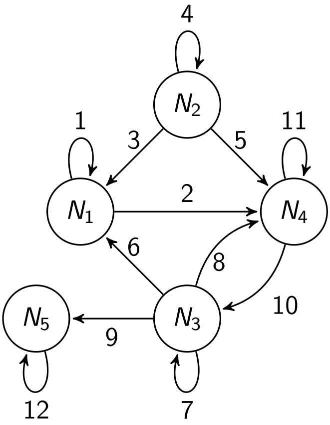
$\Rightarrow$ the Gershgorin circle theorem is too pessimistic...

Weighted directed graph representation of matrices

Define a directed graph from the nonzero entries of a matrix $A = (a_{ik})$:

- Nodes: $\mathcal{N} = \{N_i\}_{i=1\ldots n}$
- Directed edges: $\mathcal{E} = \{\overrightarrow{N_k N_l} | a_{kl} \neq 0\}$
- Matrix entries $\equiv$ weights of directed edges

$$A = \begin{pmatrix} 1. & 0. & 0. & 2. & 0. \\ 3. & 4. & 0. & 5. & 0. \\ 6. & 0. & 7. & 8. & 9. \\ 0. & 0. & 10. & 11. & 0. \\ 0. & 0. & 0. & 0. & 12. \end{pmatrix}$$



- 1:1 equivalence between matrices and weighted directed graphs
- Convenient e.g. for sparse matrices

Reducible and irreducible matrices

Definition A is *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

A is *irreducible* if it is not reducible.

Theorem (Varga, Th. 1.17): A is irreducible $\Leftrightarrow$ the matrix graph is connected, i.e. for each *ordered* pair (N_i, N_j) there is a path consisting of directed edges, connecting them.

Equivalently, for each i, j there is a sequence of consecutive nonzero matrix entries $a_{ik_1}, a_{k_1k_2}, a_{k_2k_3} \dots, a_{k_{r-1}k_r} a_{k_rj}$.

□

Taussky theorem (Olga Taussky, 1948)

Theorem (Varga, Th. 1.18) Let A be irreducible. Assume that the eigenvalue λ is a boundary point of the union of all the disks

$$\lambda \in \partial \bigcup_{i=1\dots n} \{\mu \in \mathbb{C} : |\mu - a_{ii}| \leq \Lambda_i\}$$

Then, all n Gershgorin circles pass through λ, i.e. for $i = 1 \dots n$,

$$|\lambda - a_{ii}| = \Lambda_i$$

Taussky theorem proof

Proof Assume λ is eigenvalue, $\mathbf{x}$ a corresponding eigenvector, normalized such that $\max_{i=1\dots n}|x_i| = |x_r| = 1$. From $A\mathbf{x} = \lambda\mathbf{x}$ it follows that

$$(\lambda - a_{rr})x_r = \sum_{\substack{j=1\dots n\\ j\neq r}} a_{rj}x_j \tag{1}$$

$$|\lambda - a_{rr}| \leq \sum_{\substack{j=1\dots n\\ j\neq r}} |a_{rj}|\cdot|x_j| \leq \sum_{\substack{j=1\dots n\\ j\neq r}} |a_{rj}| = \Lambda_r \tag{2}$$

λ is boundary point $\Rightarrow |\lambda - a_{rr}| = \Lambda_r$

$\Rightarrow$ For all $p \neq r$ with $a_{rp} \neq 0$, $|x_p| = 1$.

Due to irreducibility there is at least one such p. For this p, equation (2) is valid (with p in place of r) $\Rightarrow |\lambda - a_{pp}| = \Lambda_p$

Due to irreducibility, this is true for all $p = 1\dots n$. □

Consequences for heat example from Taussky theorem

- $B = I - D^{-1}A$
- We had $b_{ii} = 0$, $\Lambda_i = \begin{cases} \frac{1}{2}, & i = 1, n \\ 1 & i = 2 \dots n-1 \end{cases}$ $\Rightarrow$ estimate $|\lambda_i| \leq 1$
- Assume $|\lambda_i| = 1$. Then λ_i lies on the boundary of the union of the Gershgorin circles. But then it must lie on the boundary of both circles with radius $\frac{1}{2}$ and 1 around 0.
- Contradiction $\Rightarrow |\lambda_i| < 1$, $\rho(B) < 1$!

Diagonally dominant matrices

Definition Let $A = (a_{ij})$ be an $n \times n$ matrix.

- A is *diagonally dominant* if
  (i) for $i = 1 \ldots n$, $|a_{ii}| \geq \sum_{\substack{j=1\ldots n \\ j \neq i}} |a_{ij}|$
- A is *strictly diagonally dominant* (sdd) if
  (i) for $i = 1 \ldots n$, $|a_{ii}| > \sum_{\substack{j=1\ldots n \\ j \neq i}} |a_{ij}|$
- A is *irreducibly diagonally dominant* (idd) if
  (i) A is irreducible
  (ii) A is diagonally dominant –
  for $i = 1 \ldots n$, $|a_{ii}| \geq \sum_{\substack{j=1\ldots n \\ j \neq i}} |a_{ij}|$
  (iii) for at least one r, $1 \leq r \leq n$, $|a_{rr}| > \sum_{\substack{j=1\ldots n \\ j \neq r}} |a_{rj}|$

A very practical nonsingularity criterion

Theorem (Varga, Th. 1.21): Let A be strictly diagonally dominant or irreducibly diagonally dominant. Then A is nonsingular.

If in addition, $a_{ii} > 0$ is real for $i = 1 \dots n$, then all real parts of the eigenvalues of A are positive:

$$\mathrm{Re}\lambda_i > 0, \quad i = 1 \dots n$$

A very practical nonsingularity criterion, proof I

Proof:

- Assume A strictly diagonally dominant. Then the union of the Gershgorin disks does not contain 0 and $\lambda = 0$ cannot be an eigenvalue $\Rightarrow$ A is nonsingular.
- As for the real parts, the union of the disks is

$$\bigcup_{i=1\dots n} \{\mu \in \mathbb{C} : |\mu - a_{ii}| \leq \Lambda_i\}$$

and $\mathrm{Re}\mu$ must be larger than zero if μ should be contained.

A very practical nonsingularity criterion, proof I

- Assume A irreducibly diagonally dominant. Then, if 0 is an eigenvalue, it sits on the boundary of one of the Gershgorin disks.

  By Taussky theorem, we have $|a_{ii}| = \Lambda_i$ for all $i = 1 \dots n$.

  This is a contradiction as by definition there is at least one i such that $|a_{ii}| > \Lambda_i$

- Assume $a_{ii} > 0$, real. All real parts of the eigenvalues must be ≥ 0.

  Therefore, if a real part is 0, it lies on the boundary of at least one disk.

  By Taussky theorem it must be contained at the same time in the boundary of all the disks and in the imaginary axis.

  This contradicts the fact that there is at least one disk which does not touch the imaginary axis as by definition there is at least one i such that $|a_{ii}| > \Lambda_i$ $\square$

Corollary

Theorem: If A is complex hermitian or real symmetric, sdd or idd, with positive diagonal entries, it is positive definite.

Proof: All eigenvalues of A are real, and due to the nonsingularity criterion, they must be positive, so A is positive definite.

□

Heat conduction matrix

$$A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix}$$

- A is idd $\Rightarrow$ A is nonsingular
- $\mathrm{diag} A$ is positive real $\Rightarrow$ eigenvalues of A have positive real parts
- A is real, symmetric $\Rightarrow$ A is positive definite

Perron-Frobenius Theorem (1912/1907)

Definition: A real n-vector $\mathbf{x}$ is

- positive ($\mathbf{x} > 0$) if all entries of $\mathbf{x}$ are positive
- nonnegative ($\mathbf{x} \geq 0$) if all entries of $\mathbf{x}$ are nonnegative

Definition: A real $n \times n$ matrix A is

- positive ($A > 0$) if all entries of A are positive
- nonnegative ($A \geq 0$) if all entries of A are nonnegative

Theorem(Varga, Th. 2.7) Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.
(ii) To $\rho(A)$ there corresponds a positive eigenvector $\mathbf{x} > 0$.
(iii) $\rho(A)$ increases when any entry of A increases.
(iv) $\rho(A)$ is a simple eigenvalue of A.

Proof: See Varga. □

Perron-Frobenius for general nonnegative matrices

Each $n \times n$ matrix can be brought to the normal form

$$PAP^T = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & R_{mm} \end{pmatrix}$$

where for $j = 1 \dots m$, either R_{jj} irreducible or $R_{jj} = (0)$.

Theorem(Varga, Th. 2.20) Let $A \geq 0$ be an $n \times n$ matrix. Then

(i) A has a nonnegative eigenvalue equal to its spectral radius $\rho(A)$. This eigenvalue is positive unless A is reducible and its normal form is strictly upper triangular

(ii) To $\rho(A)$ there corresponds a nonzero eigenvector $\mathbf{x} \geq 0$.

(iii) $\rho(A)$ does not decrease when any entry of A increases.

Proof: See Varga; $\sigma(A) = \bigcup_{j=1}^{m} \sigma(R_{jj})$, apply irreducible Perron-Frobenius to R_{jj}. $\square$

Theorem on Jacobi matrix

Theorem: Let A be sdd or idd, and D its diagonal. Then

$$\rho(|I - D^{-1}A|) < 1$$

Proof: Let $B = (b_{ij}) = I - D^{-1}A$. Then

$$b_{ij} = \begin{cases} 0, & i = j \\ -\frac{a_{ij}}{a_{ii}}, & i \neq j \end{cases}$$

If A is sdd, then for $i = 1 \dots n$,

$$\sum_{j=1\dots n} |b_{ij}| = \sum_{\substack{j=1\dots n \\ j\neq i}} |\frac{a_{ij}}{a_{ii}}| = \frac{\Lambda_i}{|a_{ii}|} < 1$$

Therefore, $\rho(|B|) < 1$.

Theorem on Jacobi matrix II

If A is idd, then for $i = 1 \dots n$,

$$\sum_{j=1\dots n} |b_{ij}| = \sum_{\substack{j=1\dots n\\ j\neq i}} |\frac{a_{ij}}{a_{ii}}| = \frac{\Lambda_i}{|a_{ii}|} \leq 1$$

$$\sum_{j=1\dots n} |b_{rj}| = \frac{\Lambda_r}{|a_{rr}|} < 1 \text{ for at least one } r$$

Therefore, $\rho(|B|) <= 1$. Assume $\rho(|B|) = 1$. By Perron-Frobenius, 1 is an eigenvalue. As it is in the union of the Gershgorin disks, for some i,

$$|\lambda| = 1 \leq \frac{\Lambda_i}{|a_{ii}|} \leq 1$$

and it must lie on the boundary of this union. By Taussky then one has for all i

$$|\lambda| = 1 \leq \frac{\Lambda_i}{|a_{ii}|} = 1$$

which contradicts the idd condition. □

Jacobi method convergence

Corollary: Let A be sdd or idd, and D its diagonal. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then $\rho(I - D^{-1}A) < 1$, i.e. the Jacobi method converges.

Proof In this case, $|B| = B$ $\square$.

Regular splittings

- $A = M - N$ is a regular splitting if
  - M is nonsingular
  - M^{-1}, N are nonnegative, i.e. have nonnegative entries
- Regard the iteration $u_{k+1} = M^{-1}Nu_k + M^{-1}b$.
- We have $I - M^{-1}A = M^{-1}N$.

Convergence theorem for regular splitting

Theorem: Assume A is nonsingular, $A^{-1} \geq 0$, and $A = M - N$ is a regular splitting. Then $\rho(M^{-1}N) < 1$.

Proof: Let $G = M^{-1}N$. Then $A = M(I - G)$, therefore $I - G$ is nonsingular.

In addition

$$A^{-1}N = (M(I - M^{-1}N))^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N = (I - G)^{-1}G$$

By Perron-Frobenius (for general matrices), $\rho(G)$ is an eigenvalue with a nonnegative eigenvector $\mathbf{x}$. Thus,

$$0 \leq A^{-1}N\mathbf{x} = \frac{\rho(G)}{1 - \rho(G)}\mathbf{x}$$

Therefore $0 \leq \rho(G) \leq 1$.
As $I - G$ is nonsingular, $\rho(G) < 1$. $\square$

Convergence rate comparison

Corollary: $\rho(M^{-1}N) = \frac{\tau}{1+\tau}$ where $\tau = \rho(A^{-1}N)$.

Proof: Rearrange $\tau = \frac{\rho(G)}{1-\rho(G)}$ $\square$

Corollary: Let $A \geq 0$, $A = M_1 - N_1$ and $A = M_2 - N_2$ be regular splittings. If $N_2 \geq N_1 \geq 0$, then $1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1)$.

Proof: $\tau_2 = \rho(A^{-1}N_2) \geq \rho(A^{-1}N_1) = \tau_1$

But $\frac{\tau}{1+\tau}$ is strictly increasing. $\square$

M-Matrix definition

Definition Let A be an $n \times n$ real matrix. A is called M-Matrix if

(i) $a_{ij} \leq 0$ for $i \neq j$

(ii) A is nonsingular

(iii) $A^{-1} \geq 0$

Corollary: If A is an M-Matrix, then $A^{-1} > 0 \Leftrightarrow A$ is irreducible.

Proof: See Varga. □

Main practical M-Matrix criterion

Corollary: Let A be sdd or idd. Assume that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. Then A is an M-Matrix.

Proof: We know that A is nonsingular, but we have to show $A^{-1} \geq 0$.

- Let $B = I - D^{-1}A$. Then $\rho(B) < 1$, therefore $I - B$ is nonsingular.
- We have for $k > 0$:

$$I - B^{k+1} = (I - B)(I + B + B^2 + \cdots + B^k)$$
$$(I - B)^{-1}(I - B^{k+1}) = (I + B + B^2 + \cdots + B^k)$$

The left hand side for $k \to \infty$ converges to $(I - B)^{-1}$, therefore

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k$$

As $B \geq 0$, we have $(I - B)^{-1} = A^{-1}D \geq 0$. As $D > 0$ we must have $A^{-1} \geq 0$. □

Application

Let A be an M-Matrix. Assume $A = D - E - F$.

- Jacobi method: $M = D$ is nonsingular, $M^{-1} \geq 0$. $N = E + F$ nonnegative $\Rightarrow$ convergence
- Gauss-Seidel: $M = D - E$ is an M-Matrix as $A \leq M$ and M has non-positive off-digonal entries. $N = F \geq 0$. $\Rightarrow$ convergence
- Comparison: $N_J \geq N_{GS} \Rightarrow$ Gauss-Seidel converges faster.
- More general: Block Jacobi, Block Gauss Seidel etc.

Intermediate Summary

- Given some matrix, we now have some nice recipies to establish nonsingularity and iterative method convergence:
- **Check if the matrix is irreducible.**
  This is mostly the case for elliptic and parabolic PDEs.
- **Check if the matrix is strictly or irreducibly diagonally dominant**.
  If yes, it is in addition nonsingular.
- **Check if main diagonal entries are positive and off-diagonal entries are nonpositive.**
  If yes, in addition, the matrix is an M-Matrix, its inverse is nonnegative, and elementary iterative methods converge.
- These critera do not depend on the symmetry of the matrix!

Example: 1D finite difference matrix:

$$
Au = \begin{pmatrix}
\alpha + \frac{1}{h} & -\frac{1}{h} & & & & & \\
-\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\
& -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\
& \ddots & \ddots & \ddots & \ddots & & \\
& & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\
& & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\
& & & & -\frac{1}{h} & \frac{1}{h} + \alpha
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix}
= f =
\begin{pmatrix} \alpha v_1 \\ hf_2 \\ hf_3 \\ \vdots \\ hf_{N-2} \\ hf_{N-1} \\ \alpha v_n \end{pmatrix}
$$

- idd
- positive main diagonal entries, nonpositive off-diagonal entries

$\Rightarrow$ A is nonsingular, has the M-property, and we can e.g. apply the Jacobi and Gauss-Seidel iterative method to solve it (ok, in 1D we already know this is a bad idea . . .).

$\Rightarrow$ for $f \geq 0$ and $v \geq 0$ it follows that $u \geq 0$.
$\equiv$ heating and positive environment temperatures cannot lead to negative temperatures in the interior.