Scientific Computing WS 2017/2018

Lecture 18

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Weak formulation of homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ (here, $\operatorname{tr} u = 0$) such that

$$\int_\Omega \lambda \nabla u \nabla v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x} \; \forall v \in H_0^1(\Omega)$$

- Then,

$$a(u, v) := \int_\Omega \lambda \nabla u \nabla v \, d\mathbf{x}$$

  is a self-adjoint bilinear form defined on the Hilbert space $H_0^1(\Omega)$.
- It is bounded due to Cauchy-Schwarz:

$$|a(u, v)| = |\lambda| \cdot |\int_\Omega \nabla u \nabla v \, d\mathbf{x}| \leq ||u||_{H_0^1(\Omega)} \cdot ||v||_{H_0^1(\Omega)}$$

- $f(v) = \int_\Omega fv \, d\mathbf{x}$ is a linear functional on $H_0^1(\Omega)$. For Hilbert spaces V the dual space V' (the space of linear functionals) can be identified with the space itself.

The Lax-Milgram lemma

Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive, i.e.

$$\exists \alpha > 0 : \forall u \in V, a(u,u) \geq \alpha ||u||_V^2.$$

Then the problem: find $u \in V$ such that

$$a(u,v) = f(v) \; \forall v \in V$$

admits one and only one solution with an a priori estimate

$$||u||_V \leq \frac{1}{\alpha} ||f||_{V'}$$

The Galerkin method II

- Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive with coercivity constant α, and continuity constant γ.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v)\ \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h)\ \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

Céa's lemma

- What is the connection between u and u_h ?
- Let $v_h \in V_h$ be arbitrary. Then

$$\begin{aligned}
\alpha||u-u_h||^2 &\leq a(u-u_h, u-u_h) \quad \text{(Coercivity)} \\
&= a(u-u_h, u-v_h) + a(u-u_h, v_h-u_h) \\
&= a(u-u_h, u-v_h) \quad \text{(Galerkin Orthogonality)} \\
&\leq \gamma||u-u_h|| \cdot ||u-v_h|| \quad \text{(Boundedness)}
\end{aligned}$$

- As a result

$$||u-u_h|| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u-v_h||$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace V_h.

From the Galerkin method to the matrix equation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of V_h.
- Then, we have the representation $u_h = \sum_{j=1}^n u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

  it is actually sufficient to require

$$
\begin{aligned}
a(u_h, \phi_i) &= f(\phi_i) \; (i = 1 \dots n) \\
a\left(\sum_{j=1}^n u_j \phi_j, \phi_i\right) &= f(\phi_i) \; (i = 1 \dots n) \\
\sum_{j=1}^n a(\phi_j, \phi_i) u_j &= f(\phi_i) \; (i = 1 \dots n) \\
AU &= F
\end{aligned}
$$

  with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j)$, $F = (f_i), f_i = F(\phi_i)$, $U = (u_i)$.
- Matrix dimension is $n \times n$. Matrix sparsity ?

Matrix properties

- $a(\cdot, \cdot)$ symmetric bilinear form $\Rightarrow$ A symmetric
- Coercivity $\Rightarrow$ A positive definite

Barycentric coordinates

- Let K be a simplex.
- Functions λ_i ($i = 0 \dots d$):

$$\lambda_i : \mathbb{R}^d \to \mathbb{R}$$
$$x \mapsto \lambda_i(x) = 1 - \frac{(x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i}$$

where a_j is any vertex of K situated in F_i.

- For $x \in K$, one has

$$\begin{aligned} 1 - \frac{(x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} &= \frac{(a_j - a_i) \cdot \mathbf{n}_i - (x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} \\ &= \frac{(a_j - x) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} = \frac{\mathrm{dist}(x, F_i)}{\mathrm{dist}(a_i, F_i)} \\ &= \frac{\mathrm{dist}(x, F_i)|F_i|/d}{\mathrm{dist}(a_i, F_i)|F_i|/d} \\ &= \frac{\mathrm{dist}(x, F_i)|F_i|}{|K|} \end{aligned}$$

i.e. $\lambda_i(x)$ is the ratio of the volume of the simplex $K_i(x)$ made up of x and the vertices of F_i to the volume of K.

Polynomial space $\mathbb{P}_k$

- Space of polynomials in $x_1 \dots x_d$ of total degree $\leq k$ with real coefficients $\alpha_{i_1 \dots i_d}$:

$$\mathbb{P}_k = \left\{ p(x) = \sum_{\substack{0 \leq i_1 \dots i_d \leq k \\ i_1 + \dots + i_d \leq k}} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} \right\}$$

- Dimension:

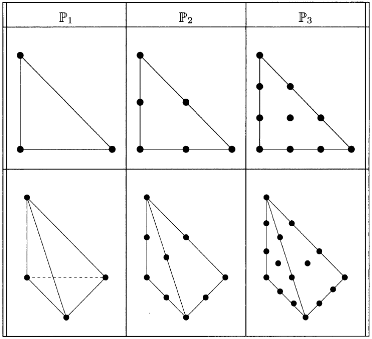
$$\dim \mathbb{P}_k = \binom{d+k}{k} = \begin{cases} k+1, & d=1 \\ \frac{1}{2}(k+1)(k+2), & d=2 \\ \frac{1}{6}(k+1)(k+2)(k+3), & d=3 \end{cases}$$

$$\dim \mathbb{P}_1 = d+1$$

$$\dim \mathbb{P}_2 = \begin{cases} 3, & d=1 \\ 6, & d=2 \\ 10, & d=3 \end{cases}$$

$\mathbb{P}_k$ simplex finite elements

- K: simplex spanned by $a_0 \dots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_k$, such that $s = \dim P_k$
- For $0 \leq i_0 \dots i_d \leq k$, $i_0 + \dots + i_d = k$, let the set of nodes be defined by the points $a_{i_1 \dots i_d;k}$ with barycentric coordinates $(\frac{i_0}{k} \dots \frac{i_d}{k})$. Define Σ by $\sigma_{i_1 \dots i_d;k}(p) = p(a_{i_1 \dots i_d;k})$.



Conformal triangulations

- Let $\mathcal{T}_h$ be a subdivision of the polygonal domain $\Omega \subset \mathbb{R}^d$ into non-intersecting compact simplices K_m, $m = 1 \dots n_e$:

$$\overline{\Omega} = \bigcup_{m=1}^{n_e} K_m$$

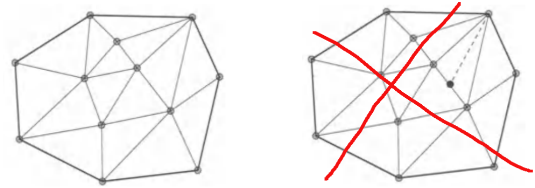
- Each simplex can be seen as the image of a affine transformation of a reference (e.g. unit) simplex $\widehat{K}$:

$$K_m = T_m(\widehat{K})$$

- We assume that it is conformal, i.e. if K_m, K_n have a $d-1$ dimensional intersection $F = K_m \cap K_n$, then there is a face $\widehat{F}$ of $\widehat{K}$ and renumberings of the vertices of K_n, K_m such that $F = T_m(\widehat{F}) = T_n(\widehat{F})$ and $T_m|_{\widehat{F}} = T_n|_{\widehat{F}}$

Conformal triangulations II

- $d = 1$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex
- $d = 2$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge



- $d = 3$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge or a common face
- Delaunay triangulations are conformal

H^1-Conformal approximation using Lagrangian finite elemenents

- Approximation space V_h with zero jumps at element faces:

$$V_h = \{v_h \in C^0(\Omega) : \forall m, v_h|_{K_m} \in \mathbb{P}^k\}$$

- $\Rightarrow$ Zero jumps at interfaces:
  $\forall n, m, K_m \cap K_n \neq 0 \Rightarrow (v_h|_{K_m})_{K_m \cap K_n} = (v_h|_{K_n})_{K_m \cap K_n}$
- $\Rightarrow V_h \subset H^1(\Omega)$.

Zero jump at interfaces with Lagrangian finite elements

- Assume geometrically conformal mesh
- Assume all faces of $\widehat{K}$ have the same number of nodes s^∂
- For any face $F = K_1 \cap K_2$ there are renumberings of the nodes of K_1 and K_2 such that for $i = 1 \dots s^\partial$, $a_{K_1,i} = a_{K_2,i}$
- Then, $v_h|_{K_1}$ and $v_h|_{K_2}$ match at the interface $K_1 \cap K_2$ if and only if they match at the common nodes

$$v_h|_{K_1}(a_{K_1,i}) = v_h|_{K_2}(a_{K_2,i}) \quad (i = 1 \dots s^\partial)$$

Global degrees of freedom

- Let $\{a_1 \dots a_N\} = \bigcup_{K \in \mathcal{T}_h} \{a_{K,1} \dots a_{K,s}\}$
- Degree of freedom map

$$j : \mathcal{T}_h \times \{1 \dots s\} \to \{1 \dots N\}$$
$$(K, m) \mapsto j(K, m) \text{ the global degree of freedom number}$$

- Global shape functions $\phi_1, \dots, \phi_N \in W_h$ defined by

$$\phi_i|_K(a_{K,m}) = \begin{cases} \delta_{mn} & \text{if } \exists n \in \{1 \dots s\} : j(K, n) = i \\ 0 & \text{otherwise} \end{cases}$$

- Global degrees of freedom $\gamma_1, \dots, \gamma_N : V_h \to \mathbb{R}$ defined by

$$\gamma_i(v_h) = v_h(a_i)$$

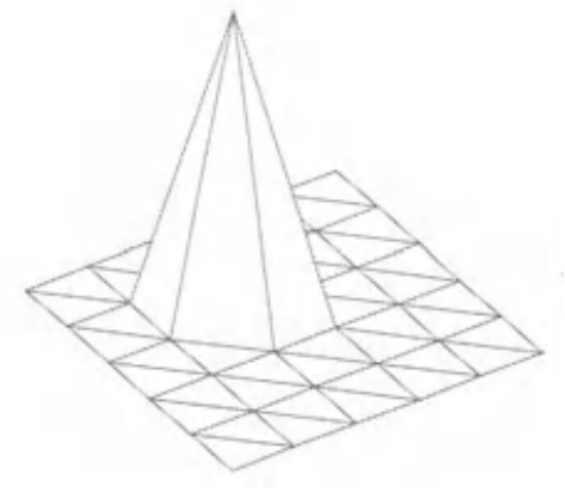
Lagrange finite element basis

- $\{\phi_1, \dots, \phi_N\}$ is a basis of V_h, and $\gamma_1 \dots \gamma_N$ is a basis of $\mathcal{L}(V_h, \mathbb{R})$.

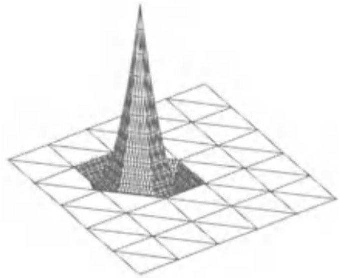
Proof:

- $\{\phi_1, \dots, \phi_N\}$ are linearly independent: if $\sum_{j=1}^N \alpha_j \phi_j = 0$ then evaluation at $a_1 \dots a_N$ yields that $\alpha_1 \dots \alpha_N = 0$.
- Let $v_h \in V_h$. It is single valued in $a_1 \dots a_N$. Let $w_h = \sum_{j=1}^N v_h(a_j)\phi_j$. Then for all $K \in \mathcal{T}_h$, $v_h|_K$ and $w_h|_K$ coincide in the local nodes $a_{K,1} \dots a_{K,2}$, and by unisolvence, $v_h|_K = w_h|_K$.

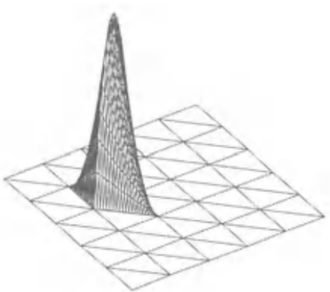
P^1 global shape functions



P^2 global shape functions



Node based



Edge based

Affine transformation estimates I

- $\widehat{K}$: reference element
- Let $K \in \mathcal{T}_h$. Affine mapping:

$$\begin{aligned} T_K : \widehat{K} &\to K \\ \widehat{x} &\mapsto J_K \widehat{x} + b_K \end{aligned}$$

  with $J_K \in \mathbb{R}^{d,d}, b_K \in \mathbb{R}^d$, J_K nonsingular
- Diameter of K: $h_K = \max_{x_1, x_2 \in K} ||x_1 - x_2||$
- ρ_K diameter of largest ball that can be inscribed into K
- $\sigma_K = \frac{h_K}{\rho_K}$: local shape regularity

Shape regularity

- Now we discuss a family of meshes $\mathcal{T}_h$ for $h \to 0$. We want to estimate global interpolation errors and see how they possibly diminish
- For given $\mathcal{T}_h$, assume that $h = \max_{K\in\mathcal{T}_h} h_j$
- A family of meshes is called *shape regular* if

$$\forall h, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0$$

- In 1D, $\sigma_K = 1$
- In 2D, $\sigma_K \leq \frac{2}{\sin \theta_K}$ where θ_K is the smallest angle

Error estimates for homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x} \; \forall v \in H_0^1(\Omega)$$

Then, $\lim_{h\to 0} ||u - u_h||_{1,\Omega} = 0$. If $u \in H^2(\Omega)$ (e.g. on convex domains) then

$$||u - u_h||_{1,\Omega} \leq ch|u|_{2,\Omega}$$
$$||u - u_h||_{0,\Omega} \leq ch^2|u|_{2,\Omega}$$

Under certain conditions (convex domain, smooth coefficients) one has

$$||u - u_h||_{0,\Omega} \leq ch|u|_{1,\Omega}$$

("Aubin-Nitsche-Lemma")

H^2-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if Ω has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients (λ) are discontinuos
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simuations
  - Deterioration of convergence rate
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretizatiom mesh

More complicated integrals

- Assume non-constant right hand side f, space dependent heat conduction coefficient κ.
- Right hand side integrals

$$f_i = \int_K f(x)\lambda_i(x)\ d\mathbf{x}$$

- P^1 stiffness matrix elements

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ d\mathbf{x}$$

- P^k stiffness matrix elements created from higher order ansatz functions

Quadrature rules

- *Quadrature rule*:

$$\int_K g(x)\ d\mathbf{x} \approx |K| \sum_{l=1}^{l_q} \omega_l g(\xi_l)$$

- ξ_l: *nodes*, *Gauss points*
- ω_l: *weights*
- The largest number k such that the quadrature is exact for polynomials of order k is called *order* k_q of the quadrature rule, i.e.

$$\forall k \leq k_q, \forall p \in \mathbb{P}^k \int_K p(x)\ d\mathbf{x} = |K| \sum_{l=1}^{l_q} \omega_l p(\xi_l)$$

- Error estimate:

$$\forall \phi \in \mathcal{C}^{k_q+1}(K), \left| \frac{1}{|K|} \int_K \phi(x)\ d\mathbf{x} - \sum_{l=1}^{l_q} \omega_l g(\xi_l) \right| \leq c h_K^{k_q+1} \sup_{x \in K, |\alpha| = k_q+1} |\partial^\alpha \phi(x)|$$

Some common quadrature rules

Nodes are characterized by the barycentric coordinates

d	k_q	l_q	Nodes	Weights
1	1	1	$(\frac{1}{2}, \frac{1}{2})$	1
	1	2	$(1,0),(0,1)$	$\frac{1}{2}, \frac{1}{2}$
	3	2	$(\frac{1}{2}+\frac{\sqrt{3}}{6}, \frac{1}{2}-\frac{\sqrt{3}}{6}),(\frac{1}{2}-\frac{\sqrt{3}}{6}, \frac{1}{2}+\frac{\sqrt{3}}{6})$	$\frac{1}{2}, \frac{1}{2}$
	5	3	$(\frac{1}{2},),(\frac{1}{2}+\sqrt{\frac{3}{20}}, \frac{1}{2}-\sqrt{\frac{3}{20}}),(\frac{1}{2}-\sqrt{\frac{3}{20}}, \frac{1}{2}+\sqrt{\frac{3}{20}})$	$\frac{8}{18}, \frac{5}{18}, \frac{5}{18}$
2	1	1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1
	1	3	$(1,0,0),(0,1,0),(0,0,1)$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	2	3	$(\frac{1}{2}, \frac{1}{2}, 0),(\frac{1}{2}, 0, \frac{1}{2}),(0, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	3	4	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}),(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}),(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}),(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}),$	$-\frac{9}{16}, \frac{25}{48}, \frac{25}{48}, \frac{25}{48}$
3	1	1	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1
	1	4	$(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$
	2	4	$(\frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5+3\sqrt{5}}{20})\dots$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$

Matching of approximation order and quadrature order

- “Variational crime”: instead of

  $$a(u_h, v_h) = f(v_h)\ \forall v_h \in V_h$$

  we solve

  $$a_h(u_h, v_h) = f_h(v_h)\ \forall v_h \in V_h$$

  where a_h, f_h are derived from their exact counterparts by quadrature
- For P^1 finite elements, zero order quadrature for volume integrals and first order quadrature for surface intergals is sufficient to keep the convergence order estimates stated before
- The rule of thumb for the volume quadrature is that the highest order terms must be evaluated exactly if the coefficients of the PDE are constant.

Practical realization of integrals

- Integral over barycentric coordinate function

$$\int_K \lambda_i(x)\ d\mathbf{x} = \frac{1}{3}|K|$$

- Right hand side integrals. Assume $f(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$f_i = \int_K f(x)\lambda_i(x)\ d\mathbf{x} \approx \frac{1}{3}|K|f(a_i)$$

- Integral over space dependent heat conduction coefficient: Assume $\kappa(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ d\mathbf{x} = \frac{1}{3}(\kappa(a_0)+\kappa(a_1)+\kappa(a_2)) \int_K \nabla\lambda_i\ \nabla\lambda_j\ d\mathbf{x}$$

Stiffness matrix for Laplace operator for P1 FEM

- Element-wise calculation:

$$a_{ij} = a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_i \nabla\phi_j \; d\mathbf{x} = \int_\Omega \sum_{K\in\mathcal{T}_h} \nabla\phi_i|_K \nabla\phi_j|_K \; d\mathbf{x}$$

- Standard assembly loop:

for $i, j = 1 \dots N$ **do**
  set $a_{ij} = 0$
end
for $K \in \mathcal{T}_h$ **do**
  for $m,n=0\dots d$ **do**

$$s_{mn} = \int_K \nabla\lambda_m \nabla\lambda_n \; d\mathbf{x}$$

$$a_{j_{dof}(K,m),j_{dof}(K,n)} = a_{j_{dof}(K,m),j_{dof}(K,n)} + s_{mn}$$

  end
end

- Local stiffness matrix:

$$S_K = (s_{K;m,n}) = \int_K \nabla\lambda_m \nabla\lambda_n \; d\mathbf{x}$$

Local stiffness matrix calculation for P1 FEM

- $a_0 \dots a_d$: vertices of the simplex K, $a \in K$.
- Barycentric coordinates: $\lambda_j(a) = \frac{|K_j(a)|}{|K|}$
- For indexing modulo d+1 we can write

$$|K| = \frac{1}{d!} \det \left(a_{j+1} - a_j, \dots a_{j+d} - a_j \right)$$
$$|K_j(a)| = \frac{1}{d!} \det \left(a_{j+1} - a, \dots a_{j+d} - a \right)$$

- From this information, we can calculate explicitely$\nabla \lambda_j(x)$ (which are constant vectors due to linearity) and the corresponding entries of the local stiffness

$$s_{ij} = \int_K \nabla \lambda_i \nabla \lambda_j \; d\mathbf{x}$$

Local stiffness matrix calculation for P1 FEM in 2D

- $a_0 = (x_0, y_0) \dots a_d = (x_2, y_2)$: vertices of the simplex K, $a = (x, y) \in K$.
- Barycentric coordinates: $\lambda_j(x, y) = \frac{|K_j(x,y)|}{|K|}$
- For indexing modulo d+1 we can write

$$
\begin{aligned}
|K| &= \frac{1}{2} \det \begin{pmatrix} x_{j+1} - x_j & x_{j+2} - x_j \\ y_{j+1} - y_j & y_{j+2} - y_j \end{pmatrix} \\
|K_j(x, y)| &= \frac{1}{2} \det \begin{pmatrix} x_{j+1} - x & x_{j+2} - x \\ y_{j+1} - y & y_{j+2} - y \end{pmatrix}
\end{aligned}
$$

- Therefore, we have

$$
\begin{aligned}
|K_j(x, y)| &= \frac{1}{2} \left((x_{j+1} - x)(y_{j+2} - y) - (x_{j+2} - x)(y_{j+1} - y) \right) \\
\partial_x |K_j(x, y)| &= \frac{1}{2} \left((y_{j+1} - y) - (y_{j+2} - y) \right) = \frac{1}{2}(y_{j+1} - y_{j+2}) \\
\partial_y |K_j(x, y)| &= \frac{1}{2} \left((x_{j+2} - x) - (x_{j+1} - x) \right) = \frac{1}{2}(x_{j+2} - x_{j+1})
\end{aligned}
$$

Local stiffness matrix calculation for P1 FEM in 2D II

- $$s_{ij} = \int_K \nabla\lambda_i \nabla\lambda_j \; d\mathbf{x} = \frac{|K|}{4|K|^2} \left(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}\right) \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$
- So, let $V = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix}$
- Then

$$x_1 - x_2 = V_{00} - V_{01}$$
$$y_1 - y_2 = V_{10} - V_{11}$$

and

$$2|K| \; \nabla\lambda_0 = \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} V_{10} - V_{11} \\ V_{01} - V_{00} \end{pmatrix}$$
$$2|K| \; \nabla\lambda_1 = \begin{pmatrix} y_2 - y_0 \\ x_0 - x_2 \end{pmatrix} = \begin{pmatrix} V_{11} \\ -V_{01} \end{pmatrix}$$
$$2|K| \; \nabla\lambda_2 = \begin{pmatrix} y_0 - y_1 \\ x_1 - x_0 \end{pmatrix} = \begin{pmatrix} -V_{10} \\ V_{00} \end{pmatrix}$$

Degree of freedom map representation for P1 finite elements

- List of global nodes $a_0 \dots a_N$: two dimensional array of coordinate values with N rows and d columns
- Local-global degree of freedom map: two-dimensional array C of index values with N_{el} rows and $d+1$ columns such that $C(i,m) = j_{dof}(K_i, m)$.
- The mesh generator triangle generates this information directly

Practical realization of boundary conditions

- Robin boundary value problem

$$
\begin{aligned}
-\nabla \cdot \kappa \nabla u &= f \quad \text{in } \Omega \\
\kappa \nabla u + \alpha(u - g) &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

- Weak formulation: search $u \in H^1(\Omega)$ such that

$$
\int_\Omega \kappa \nabla u \nabla v \, d\mathbf{x} + \int_{\partial\Omega} \alpha uv \; ds = \int_\Omega fv \, d\mathbf{x} + \int_{\partial\Omega} \alpha gv \, ds \; \forall v \in H^1(\Omega)
$$

- In 2D, for P^1 FEM, boundary integrals can be calculated by trapezoidal rule without sacrificing approximation order
- Use Dirichlet penalty method to handle Dirichlet boundary conditions

Next lecture

Next lecture: Jan. 9, 2018.

Happy holidays!