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P1 finite elements: time dependent problems

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Lecture 22

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Recap

Global interpolation error estimate for Lagrangian finite elements, $k = 1$

- Assume $v \in H^2(\Omega)$

$$||v - \mathcal{I}_h^k v||_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} \leq ch^2|v|_{2,\Omega}$$
$$|v - \mathcal{I}_h^k v|_{1,\Omega} \leq ch|v|_{2,\Omega}$$
$$\lim_{h\to 0}\left(\inf_{v_h \in V_h^k} |v - v_h|_{1,\Omega}\right) = 0$$

- If $v \in H^2(\Omega)$ cannot be guaranteed, estimates become worse. Example: L-shaped domain.
- These results immediately can be applied in Cea's lemma.

Error estimates for homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, dx = \int_\Omega fv \, dx \;\forall v \in H_0^1(\Omega)$$

- Then, $\lim_{h\to 0} ||u - u_h||_{1,\Omega} = 0$.
- If $u \in H^2(\Omega)$ (e.g. convex domain, smooth coefficients), then

$$||u - u_h||_{1,\Omega} \leq ch|u|_{2,\Omega} \leq c'h|f|_{0,\Omega}$$
$$||u - u_h||_{0,\Omega} \leq ch^2|u|_{2,\Omega} \leq c'h^2|f|_{0,\Omega}$$

and ("Aubin-Nitsche-Lemma")

$$||u - u_h||_{0,\Omega} \leq ch|u|_{1,\Omega}$$

H^2-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if Ω has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients (λ) are discontinuos
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simuations
  - Deterioration of convergence ratw
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretizatiom mesh

Higher regularity

- If $u \in H^s(\Omega)$ for $s > 2$, convergence order estimates become even better for P^k finite elements of order $k > 1$.
- Depending on the regularity of the solution the combination of grid adaptation and higher oder ansatz functions may be successful

More complicated integrals

- Assume non-constant right hand side f, space dependent heat conduction coefficient κ.
- Right hand side integrals

$$f_i = \int_K f(x)\lambda_i(x)\ dx$$

- P^1 stiffness matrix elements

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ dx$$

- P^k stiffness matrix elements created from higher order ansatz functions

Quadrature rules

- *Quadrature rule*:

$$\int_K g(x)\ dx \approx |K| \sum_{l=1}^{l_q} \omega_l g(\xi_l)$$

- ξ_l: *nodes*, *Gauss points*
- ω_l: *weights*
- The largest number k such that the quadrature is exact for polynomials of order k is called *order* k_q of the quadrature rule, i.e.

$$\forall k \leq k_q, \forall p \in \mathbb{P}^k \int_K p(x)\ dx = |K| \sum_{l=1}^{l_q} \omega_l p(\xi_l)$$

- Error estimate:

$$\forall \phi \in \mathcal{C}^{k_q+1}(K), \left| \frac{1}{|K|} \int_K \phi(x)\ dx - \sum_{l=1}^{l_q} \omega_l g(\xi_l) \right| \leq c h_K^{k_q+1} \sup_{x \in K, |\alpha| = k_q+1} |\partial^\alpha \phi(x)|$$

Some common quadrature rules

Nodes are characterized by the barycentric coordinates

d	k_q	l_q	Nodes	Weights
1	1	1	$(\frac{1}{2}, \frac{1}{2})$	1
	1	2	$(1, 0), (0, 1)$	$\frac{1}{2}, \frac{1}{2}$
	3	2	$(\frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6}), (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$	$\frac{1}{2}, \frac{1}{2}$
	5	3	$(\frac{1}{2},), (\frac{1}{2} + \sqrt{\frac{3}{20}}, \frac{1}{2} - \sqrt{\frac{3}{20}}), (\frac{1}{2} - \sqrt{\frac{3}{20}}, \frac{1}{2} + \sqrt{\frac{3}{20}})$	$\frac{8}{18}, \frac{5}{18}, \frac{5}{18}$
2	1	1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1
	1	3	$(1, 0, 0), (0, 1, 0), (0, 0, 1)$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	2	3	$(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$
	3	4	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}), (\frac{1}{5}, \frac{3}{5}, \frac{1}{5}), (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}),$	$-\frac{9}{16}, \frac{25}{48}, \frac{25}{48}, \frac{25}{48}$
3	1	1	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1
	1	4	$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$
	2	4	$(\frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5+3\sqrt{5}}{20}) \dots$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$

Matching of approximation order and quadrature order

- "Variational crime": instead of

$$a(u_h, v_h) = f(v_h) \ \forall v_h \in V_h$$

  we solve

$$a_h(u_h, v_h) = f_h(v_h) \ \forall v_h \in V_h$$

  where a_h, f_h are derived from their exact counterparts by quadrature
- For P^1 finite elements, zero order quadrature for volume integrals and first order quadrature for surface intergals is sufficient to keep the convergence order estimates stated before
- The rule of thumb for the volume quadrature is that the highest order terms must be evaluated exactly if the coefficients of the PDE are constant.

Practical realization of integrals

- Integral over barycentric coordinate function

$$\int_K \lambda_i(x)\ dx = \frac{1}{3}|K|$$

- Right hand side integrals. Assume $f(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$f_i = \int_K f(x)\lambda_i(x)\ dx \approx \frac{1}{3}|K|f_(a_i)$$

- Integral over space dependent heat conduction coefficient: Assume $\kappa(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ dx = \frac{1}{3}(\kappa(a_0)+\kappa(a_1)+\kappa(a_2))\int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ dx$$

Practical realization of boundary conditions

- Robin boundary value problem

$$
\begin{aligned}
-\nabla \cdot \kappa \nabla u &= f \quad \text{in } \Omega \\
\kappa \nabla u + \alpha(u - g) &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

- Weak formulation: search $u \in H^1(\Omega)$ such that

$$
\int_\Omega \kappa \nabla u \nabla v \, dx + \int_{\partial\Omega} \alpha u v \, ds = \int_\Omega f v \, dx + \int_{\partial\Omega} \alpha g v \, ds \; \forall v \in H^1(\Omega)
$$

- In 2D, for P^1 FEM, boundary integrals can be calculated by trapezoidal rule without sacrificing approximation order

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Time dependent problems

Time dependent Robin boundary value problem

- Choose final time $T > 0$. Regard functions $(x, t) \to \mathbb{R}$.

$$\begin{aligned} \partial_t u - \nabla \cdot \kappa \nabla u = f \quad & \text{in } \Omega \times [0, T] \\ \kappa \nabla u + \alpha(u - g) = 0 \quad & \text{on } \partial\Omega \times [0, T] \\ u(x, 0) = u_0(x) \quad & \text{in} \Omega \end{aligned}$$

- This is an initial boundary value problem
- This problem has a weak formulation in the Sobolev space $L^2\left([0, T], H^1(\Omega)\right)$, which then allows for a Galerkin approximation in a corresponding subspace
- We will proceed in a simpler manner: first, perform a finite difference discretization in time, then perform a finite element (finite volume) discretization in space.
    - Rothe method: first discretize in time, then in space
    - Method of lines: first discretize in space, get a huge ODE system

Time discretization

- Choose time discretization points $0 = t_0 < t_1 \dots < t_N = T$, let $\tau_i = t_i - t_{i-1}$
  For $i = 1 \dots N$, solve

$$\frac{u_i - u_{i-1}}{\tau_i} - \nabla \cdot \kappa \nabla u_\theta = f \quad \text{in } \Omega \times [0, T]$$
$$\kappa \nabla u_\theta + \alpha(u_\theta - g) = 0 \quad \text{on } \partial\Omega \times [0, T]$$

  where $u_\theta = \theta u_i + (1-\theta) u_{i-1}$
- $\theta = 1$: backward (implicit) Euler method
- $\theta = \frac{1}{2}$: Crank-Nicolson scheme
- $\theta = 0$: forward (explicit) Euler method
- Note that the explicit Euler method does not involve the solution of a PDE system. What do we have to pay for this ?

Weak formulation

- Weak formulation: search $u \in H^1(\Omega)$ such that

$$
\begin{aligned}
\frac{1}{\tau_i}\int_\Omega u_i v \, dx + \theta \left(\int_\Omega \kappa \nabla u_i \nabla v \, dx + \int_{\partial\Omega} \alpha u_i v \, ds \right) = \\
\frac{1}{\tau_i}\int_\Omega u_{i-1} v \, dx + (1-\theta) \left(\int_\Omega \kappa \nabla u_{i-1} \nabla v \, dx + \int_{\partial\Omega} \alpha u_{i-1} v \, ds \right) \\
+ \int_\Omega f v \, dx + \int_{\partial\Omega} \alpha g v \, ds \; \forall v \in H^1(\Omega)
\end{aligned}
$$

- Matrix formulation (in case of constant coefficents, $A_i = A$)

$$
\frac{1}{\tau_i} M u_i + \theta A_i u_i = \frac{1}{\tau_i} M u_{i-1} + (1-\theta) A_i u_{i-1} + F
$$

- M: mass matrix, A: stiffness matrix