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# P1 finite elements: wrap up

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Lecture 21

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# Recap

# The Galerkin method

- Let $V$ be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and $f$ a linear functional on $V$. Assume $a$ is coercive with coercivity constant $\alpha$, and continuity constant $\gamma$.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v) \; \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of $V$
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

# Céa's lemma

- What is the connection between $u$ and $u_h$ ?
- Let $v_h \in V_h$ be arbitrary. Then

$$\begin{aligned}
\alpha ||u - u_h||^2 &\leq a(u - u_h, u - u_h) \quad \text{(Coercivity)} \\
&= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
&= a(u - u_h, u - v_h) \quad \text{(Galerkin Orthogonality)} \\
&\leq \gamma ||u - u_h|| \cdot ||u - v_h|| \quad \text{(Boundedness)}
\end{aligned}$$

- As a result

$$||u - u_h|| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u - v_h||$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace $V_h$.

# Definition of a Finite Element (Ciarlet)

Triplet $\{K, P, \Sigma\}$ where

- $K \subset \mathbb{R}^d$: compact, connected Lipschitz domain with non-empty interior
- $P$: finite dimensional vector space of functions $p : K \to \mathbb{R}^m$ (mostly, $m = 1, m = d$)
- $\Sigma = \{\sigma_1 \dots \sigma_s\} \subset \mathcal{L}(P, \mathbb{R})$: set of linear forms defined on $P$ called *local degrees of freedom* such that the mapping

$$\Lambda_\Sigma : P \to \mathbb{R}^s$$
$$p \mapsto (\sigma_1(p) \dots \sigma_s(p))$$

is bijective, i.e. $\Sigma$ is a basis of $\mathcal{L}(P, \mathbb{R})$.

# Local shape functions

- Due to bijectivity of $\Lambda_\Sigma$, for any finite element $\{K, P, \Sigma\}$, there exists a basis $\{\theta_1 \dots \theta_s\} \subset P$ such that

$$\sigma_i(\theta_j) = \delta_{ij} \quad (1 \leq i, j \leq s)$$

- Elements of such a basis are called *local shape functions*

# Unisolvence

- Bijectivity of $\Lambda_\Sigma$ is equivalent to the condition

$$\forall(\alpha_1 \dots \alpha_s) \in \mathbb{R}^s \; \exists! p \in P \text{ such that } \sigma_i(p) = \alpha_i \quad (1 \le i \le s)$$

  i.e. for any given tuple of values $a = (\alpha_1 \dots \alpha_s)$ there is a unique polynomial $p \in P$ such that $\Lambda_\Sigma(p) = a$.
- Equivalent to *unisolvence*:

$$\begin{cases} \dim P = |\Sigma| = s \\ \forall p \in P : \; \sigma_i(p) = 0 \, (i = 1 \dots s) \; \Rightarrow \; p = 0 \end{cases}$$

# Lagrange finite elements

- A finite element $\{K, P, \Sigma\}$ is called *Lagrange* finite element (or *nodal* finite element) if there exist a set of points $\{a_1 \dots a_s\} \subset K$ such that

$$\sigma_i(p) = p(a_i) \quad 1 \leq i \leq s$$

- $\{a_1 \dots a_s\}$: *nodes* of the finite element
- *nodal basis*: $\{\theta_1 \dots \theta_s\} \subset P$ such that

$$\theta_j(a_i) = \delta_{ij} \quad (1 \leq i, j \leq s)$$

# Local interpolation operator

- Let $\{K, P, \Sigma\}$ be a finite element with shape function bases $\{\theta_1 \dots \theta_s\}$. Let $V(K)$ be a normed vector space of functions $v : K \to \mathbb{R}^m$ such that
  - $P \subset V(K)$
  - The linear forms in $\Sigma$ can be extended to be defined on $V(K)$
- *local interpolation operator*

$$\begin{aligned} \mathcal{I}_K : V(K) &\to P \\ v &\mapsto \sum_{i=1}^{s} \sigma_i(v)\theta_i \end{aligned}$$

- $P$ is invariant under the action of $\mathcal{I}_K$, i.e. $\forall p \in P, \mathcal{I}_K(p) = p$.

# Local Lagrange interpolation operator

- Let $V(K) = (\mathcal{C}^0(K))^m$

$$\mathcal{I}_K : V(K) \to P$$
$$v \mapsto I_K v = \sum_{i=1}^{s} v(a_i)\theta_i$$

# Simplices

- Let $\{a_0 \dots a_d\} \subset \mathbb{R}^d$ such that the $d$ vectors $a_1 - a_0 \dots a_d - a_0$ are linearly independent. Then the convex hull $K$ of $a_0 \dots a_d$ is called *simplex*, and $a_0 \dots a_d$ are called *vertices* of the simplex.
- *Unit simplex*: $a_0 = (0...0), a_1 = (0, 1 \dots 0) \dots a_d = (0 \dots 0, 1)$.

$$K = \left\{ x \in \mathbb{R}^d : x_i \geq 0 \ (i = 1 \dots d) \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}$$

- A general simplex can be defined as an image of the unit simplex under some affine transformation
- $F_i$: face of $K$ opposite to $a_i$
- $\mathbf{n}_i$: outward normal to $F_i$

## Barycentric coordinates

- Let $K$ be a simplex.
- Functions $\lambda_i$ $(i = 0 \dots d)$:

$$\begin{aligned} \lambda_i : \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto \lambda_i(x) = 1 - \frac{(x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} \end{aligned}$$

  where $a_j$ is any vertex of $K$ situated in $F_i$.
- For $x \in K$, one has

$$\begin{aligned} 1 - \frac{(x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} &= \frac{(a_j - a_i) \cdot \mathbf{n}_i - (x - a_i) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} \\ &= \frac{(a_j - x) \cdot \mathbf{n}_i}{(a_j - a_i) \cdot \mathbf{n}_i} = \frac{\mathrm{dist}(x, F_i)}{\mathrm{dist}(a_i, F_i)} \\ &= \frac{\mathrm{dist}(x, F_i)|F_i|/d}{\mathrm{dist}(a_i, F_i)|F_i|/d} \\ &= \frac{|K_i(x)|}{|K|} \end{aligned}$$

  i.e. $\lambda_i(x)$ is the ratio of the volume of the simplex $K_i(x)$ made up of $x$ and the vertices of $F_i$ to the volume of $K$.

## Barycentric coordinates II

- $\lambda_i(a_j) = \delta_{ij}$
- $\lambda_i(x) = 0 \; \forall x \in F_i$
- $\sum_{i=0}^{d} \lambda_i(x) = 1 \; \forall x \in \mathbb{R}^d$
  (just sum up the volumes)
- $\sum_{i=0}^{d} \lambda_i(x)(x - a_i) = 0 \; \forall x \in \mathbb{R}^d$
  (due to $\sum \lambda_i(x) x = x$ and $\sum \lambda_i a_i = x$ as the vector of linear coordinate functions)
- Unit simplex:
  - $\lambda_0(x) = 1 - \sum_{i=1}^{d} x_i$
  - $\lambda_i(x) = x_i$ for $1 \leq i \leq d$

# Polynomial space $\mathbb{P}_k$

- Space of polynomials in $x_1 \dots x_d$ of total degree $\leq k$ with real coefficients $\alpha_{i_1 \dots i_d}$:

$$\mathbb{P}_k = \left\{ p(x) = \sum_{\substack{0 \leq i_1 \dots i_d \leq k \\ i_1 + \dots + i_d \leq k}} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} \right\}$$

- Dimension:

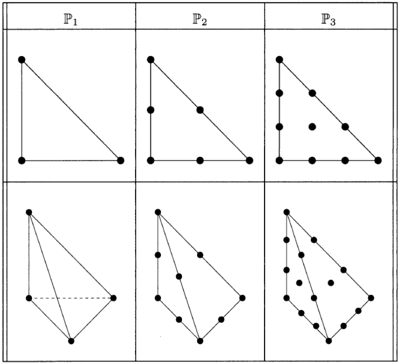
$$\dim \mathbb{P}_k = \binom{d+k}{k} = \begin{cases} k+1, & d=1 \\ \frac{1}{2}(k+1)(k+2), & d=2 \\ \frac{1}{6}(k+1)(k+2)(k+3), & d=3 \end{cases}$$

$$\dim \mathbb{P}_1 = d+1$$

$$\dim \mathbb{P}_2 = \begin{cases} 3, & d=1 \\ 6, & d=2 \\ 10, & d=3 \end{cases}$$

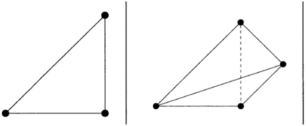
## $\mathbb{P}_k$ simplex finite elements

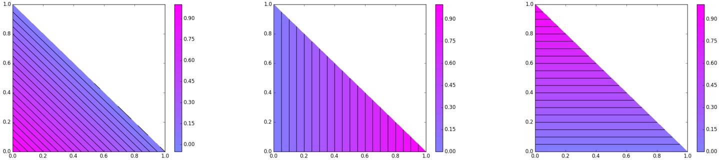
- $K$: simplex spanned by $a_0 \dots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_k$, such that $s = \dim P_k$
- For $0 \le i_0 \dots i_d \le k$, $i_0 + \dots + i_d = k$, let the set of nodes be defined by the points $a_{i_1 \dots i_d;k}$ with barycentric coordinates $(\frac{i_0}{k} \dots \frac{i_d}{k})$.
  Define $\Sigma$ by $\sigma_{i_1 \dots i_d;k}(p) = p(a_{i_1 \dots i_d;k})$.



# $\mathbb{P}_1$ simplex finite elements

- $K$: simplex spanned by $a_0 \dots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_1$, such that $s = d + 1$
- Nodes $\equiv$ vertices
- Basis functions $\equiv$ barycentric coordinates





## Conformal triangulations

- Let $\mathcal{T}_h$ be a subdivision of the polygonal domain $\Omega \subset \mathbb{R}^d$ into non-intersecting compact simplices $K_m$, $m = 1 \dots n_e$:

$$\overline{\Omega} = \bigcup_{m=1}^{n_e} K_m$$

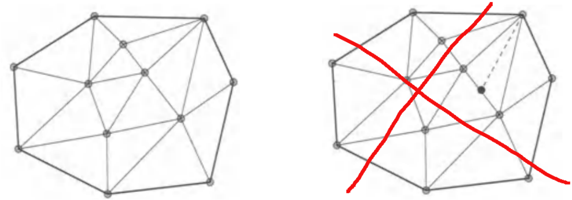
- Each simplex can be seen as the image of a affine transormation of a reference (e.g. unit) simplex $\widehat{K}$:

$$K_m = T_m(\widehat{K})$$

- We assume that it is conformal, i.e. if $K_m$, $K_n$ have a $d-1$ dimensional intersection $F = K_m \cap K_n$, then there is a face $\widehat{F}$ of $\widehat{K}$ and renumberings of the vertices of $K_n, K_m$ such that $F = T_m(\widehat{F}) = T_n(\widehat{F})$ and $T_m|_{\widehat{F}} = T_n|_{\widehat{F}}$

# Conformal triangulations II

- $d = 1$ : Each intersection $F = K_m \cap K_n$ is either empty or a common vertex
- $d = 2$ : Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge



- $d = 3$ : Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge or a common face
- Triangulations corresponding to simplicial complexes are conformal
- Delaunay triangulations are conformal

# Reference finite element

- Let $\{\widehat{P}, \widehat{K}, \widehat{\Sigma}\}$ be a fixed finite element
- Let $T_K$ be some affine transformation and $K = T_K(\widehat{K})$
- There is a linear bijective mapping $\psi_K$ between functions on $K$ and functions on $\widehat{K}$:

$$\begin{aligned}\psi_K : V(K) &\to V(\widehat{K})\\ f &\mapsto f \circ T_K\end{aligned}$$

- Let
  - $K = T_K(\widehat{K})$$
  - $P_K = \{\psi_K^{-1}(\widehat{p}); \widehat{p} \in \widehat{P}\}$,
  - $\Sigma_K = \{\sigma_{K,i}, i = 1 \ldots s : \sigma_{K,i}(p) = \widehat{\sigma}_i(\psi_K(p))\}$ Then $\{K, P_K, \Sigma_K\}$ is a finite element.

# Commutativity of interpolation and reference mapping

- $\mathcal{I}_{\hat{K}} \circ \psi_K = \psi_K \circ \mathcal{I}_K$,
  i.e. the following diagram is commutative:

$$
\begin{array}{ccc}
V(K) & \xrightarrow{\psi_K} & V(\widehat{K}) \\
\Big\downarrow \mathcal{I}_K & & \Big\downarrow \mathcal{I}_{\hat{K}} \\
P_K & \xrightarrow{\psi_K} & P_{\widehat{K}}
\end{array}
$$

## Global interpolation operator $\mathcal{I}_h$

- Let $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ be a triangulation of $\Omega$.
- Domain:

$$D(\mathcal{I}_h) = \{v \in (L^1(\Omega))^m \text{ such that } \forall K \in \mathcal{T}_h, v|_K \in V(K)\}$$

- For all $v \in D(\mathcal{I}_h)$, define $\mathcal{I}_h v$ via

$$\mathcal{I}_h v|_K = \mathcal{I}_K(v|_K) = \sum_{i=1}^{s} \sigma_{K,i}(v|_K)\theta_{K,i} \ \forall K \in \mathcal{T}_h,$$

  Assuming $\theta_{K,i} = 0$ outside of $K$, one can write

$$\mathcal{I}_h v = \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{s} \sigma_{K,i}(v|_K)\theta_{K,i},$$

  mapping $D(\mathcal{I}_h)$ to the *approximation space*

$$W_h = \{v_h \in (L^1(\Omega))^m \text{ such that } \forall K \in \mathcal{T}_h, v_h|_K \in P_K\}$$

# $H^1$-Conformal approximation using Lagrangian finite elemenents

- Let $V$ be a Banach space of functions on $\Omega$. The approximation space $W_h$ is said to be $V$-conformal if $W_h \subset V$.
- Non-conformal approximations are possible, we will stick to the conformal case.
- Conformal subspace of $W_h$ with zero jumps at element faces:

$$V_h = \{v_h \in W_h : \forall n, m, K_m \cap K_n \neq 0 \Rightarrow (v_h|_{K_m})_{K_m \cap K_n} = (v_h|_{K_n})_{K_m \cap K_n}\}$$

- Then: $V_h \subset H^1(\Omega)$

# Zero jump at interfaces with Lagrangian finite elements

- Assume geometrically conformal mesh
- Assume all faces of $\widehat{K}$ have the same number of nodes $s^\partial$
- For any face $F = K_1 \cap K_2$ there are renumberings of the nodes of $K_1$ and $K_2$ such that for $i = 1 \ldots s^\partial$, $a_{K_1,i} = a_{K_2,i}$
- Then, $v_h|_{K_1}$ and $v_h|_{K_2}$ match at the interface $K_1 \cap K_2$ if and only if they match at the common nodes

$$v_h|_{K_1}(a_{K_1,i}) = v_h|_{K_2}(a_{K_2,i}) \quad (i = 1 \ldots s^\partial)$$

# Global degrees of freedom

- Let $\{a_1 \dots a_N\} = \bigcup_{K \in \mathcal{T}_h} \{a_{K,1} \dots a_{K,s}\}$
- Degree of freedom map

$$j : \mathcal{T}_h \times \{1 \dots s\} \to \{1 \dots N\}$$
$$(K, m) \mapsto j(K, m) \text{ the global degree of freedom number}$$

- Global shape functions $\phi_1, \dots, \phi_N \in W_h$ defined by

$$\phi_i|_K(a_{K,m}) = \begin{cases} \delta_{mn} & \text{if } \exists n \in \{1 \dots s\} : j(K, n) = i \\ 0 & \text{otherwise} \end{cases}$$

- Global degrees of freedom $\gamma_1, \dots, \gamma_N : V_h \to \mathbb{R}$ defined by

$$\gamma_i(v_h) = v_h(a_i)$$

# From the Galerkin method to the matrix equation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of $V_h$.
- Then, we have the representation $u_h = \sum_{j=1}^{n} u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \ \forall v_h \in V_h$$

it is actually sufficient to require

$$\begin{aligned} a(u_h, \phi_i) &= f(\phi_i) \ (i = 1 \dots n) \\ a\left(\sum_{j=1}^{n} u_j \phi_j, \phi_i\right) &= f(\phi_i) \ (i = 1 \dots n) \\ \sum_{j=1}^{n} a(\phi_j, \phi_i) u_j &= f(\phi_i) \ (i = 1 \dots n) \\ AU &= F \end{aligned}$$

with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j),\ F = (f_i), f_i = F(\phi_i),\ U = (u_i)$.
- Matrix dimension is $n \times n$.

# Stiffness matrix calculation for Laplace operator for P1 FEM

$$
\begin{aligned}
a_{ij} = a(\phi_i, \phi_j) &= \int_\Omega \nabla\phi_i \nabla\phi_j \; dx \\
&= \int_\Omega \sum_{K\in\mathcal{T}_h} \nabla\phi_i|_K \nabla\phi_j|_K \; dx
\end{aligned}
$$

Assembly loop:
Set $a_{ij} = 0$.
For each $K \in \mathcal{T}_h$:
For each $m, n = 0 \dots d$:

$$
s_{mn} = \int_K \nabla\lambda_m \nabla\lambda_n \; dx
$$

$$
a_{j_{dof}(K,m), j_{dof}(K,n)} = a_{j_{dof}(K,m), j_{dof}(K,n)} + s_{mn}
$$

# Local stiffness matrix calculation for P1 FEM

$a_0 \dots a_d$: vertices of the simplex $K$, $a \in K$.

Barycentric coordinates: $\lambda_j(a) = \frac{|K_j(a)|}{|K|}$

For indexing modulo d+1 we can write

$$|K| = \frac{1}{d!} \det \left( a_{j+1} - a_j, \dots a_{j+d} - a_j \right)$$
$$|K_j(a)| = \frac{1}{d!} \det \left( a_{j+1} - a, \dots a_{j+d} - a \right)$$

From this information, we can calculate $\nabla \lambda_j(x)$ (which are constant vectors due to linearity) and the corresponding entries of the local stiffness matrix

$$s_{ij} = \int_K \nabla \lambda_i \nabla \lambda_j \; dx$$

# Local stiffness matrix calculation for P1 FEM in 2D II

$$s_{ij} = \int_K \nabla\lambda_i \nabla\lambda_j \, dx = \frac{|K|}{4|K|^2} \begin{pmatrix} y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1} \end{pmatrix} \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$

So, let $V = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix}$

Then

$$\begin{aligned} x_1 - x_2 &= V_{00} - V_{01} \\ y_1 - y_2 &= V_{10} - V_{11} \end{aligned}$$

and

$$\begin{aligned} 2|K| \; \nabla\lambda_0 &= \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} V_{10} - V_{11} \\ V_{01} - V_{00} \end{pmatrix} \\ 2|K| \; \nabla\lambda_1 &= \begin{pmatrix} y_2 - y_0 \\ x_0 - x_2 \end{pmatrix} = \begin{pmatrix} V_{11} \\ -V_{01} \end{pmatrix} \\ 2|K| \; \nabla\lambda_2 &= \begin{pmatrix} y_0 - y_1 \\ x_1 - x_0 \end{pmatrix} = \begin{pmatrix} -V_{10} \\ V_{00} \end{pmatrix} \end{aligned}$$

# Local stiffness matrix calculation for P1 FEM in 2D II

$$s_{ij} = \int_K \nabla\lambda_i \nabla\lambda_j \; dx = \frac{|K|}{4|K|^2} \left(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}\right) \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$

So, let $V = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix}$

Then

$$\begin{aligned} x_1 - x_2 &= V_{00} - V_{01} \\ y_1 - y_2 &= V_{10} - V_{11} \end{aligned}$$

and

$$\begin{aligned} 2|K| \; \nabla\lambda_0 &= \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} V_{10} - V_{11} \\ V_{01} - V_{00} \end{pmatrix} \\ 2|K| \; \nabla\lambda_1 &= \begin{pmatrix} y_2 - y_0 \\ x_0 - x_2 \end{pmatrix} = \begin{pmatrix} V_{11} \\ -V_{01} \end{pmatrix} \\ 2|K| \; \nabla\lambda_2 &= \begin{pmatrix} y_0 - y_1 \\ x_1 - x_0 \end{pmatrix} = \begin{pmatrix} -V_{10} \\ V_{00} \end{pmatrix} \end{aligned}$$

# Degree of freedom map representation for P1 finite elements

- List of global nodes $a_0 \dots a_N$: two dimensional array of coordinate values with $N$ rows and $d$ columns
- Local-global degree of freedom map: two-dimensional array $C$ of index values with $N_{el}$ rows and $d+1$ columns such that $C(i,m) = j_{dof}(K_i, m)$.
- The mesh generator triangle generates this information directly

# Finite element assembly loop

```
for (int icell=0; icell<ncells; icell++)
{
  // Fill matrix V
  V(0,0)= points(cells(icell,1),0)- points(cells(icell,0),0);
  V(0,1)= points(cells(icell,2),0)- points(cells(icell,0),0);

  V(1,0)= points(cells(icell,1),1)- points(cells(icell,0),1);
  V(1,1)= points(cells(icell,2),1)- points(cells(icell,0),1);

  // Compute determinant
  double det=V(0,0)*V(1,1) - V(0,1)*V(1,0);
  double invdet = 1.0/det;

  // Compute entris of local stiffness matrix
  SLocal(0,0)= invdet * (  ( V(1,0)-V(1,1) )*( V(1,0)-V(1,1) )
                          +( V(0,1)-V(0,0) )*( V(0,1)-V(0,0) ) );
  SLocal(0,1)= invdet * (  ( V(1,0)-V(1,1) )* V(1,1)          - ( V(0,1)-V(0,0) )*V(0,1) );
  SLocal(0,2)= invdet * ( -( V(1,0)-V(1,1) )* V(1,0)          + ( V(0,1)-V(0,0) )*V(0,0) );

  SLocal(1,1)=  invdet * (  V(1,1)*V(1,1) + V(0,1)*V(0,1) );
  SLocal(1,2)=  invdet * ( -V(1,1)*V(1,0) - V(0,1)*V(0,0) );

  SLocal(2,2)=  invdet * ( V(1,0)*V(1,0)+ V(0,0)*V(0,0) );

  SLocal(1,0)=SLocal(0,1);
  SLocal(2,0)=SLocal(0,2);
  SLocal(2,1)=SLocal(1,2);

  // Assemble into global stiffness matrix
  for (int i=0;i<=ndim;i++)
    for (int j=0;j<=ndim;j++)
      SGlobal(cells(icell,i),cells(icell,j))+=SLocal(i,j);
}
```

# Affine transformation estimates I

- $\widehat{K}$: reference element
- Let $K \in \mathcal{T}_h$. Affine mapping:

$$\begin{aligned} T_K : \widehat{K} &\to K \\ \widehat{x} &\mapsto J_K \widehat{x} + b_K \end{aligned}$$

  with $J_K \in \mathbb{R}^{d,d}$, $b_K \in \mathbb{R}^d$, $J_K$ nonsingular
- Diameter of $K$: $h_K = \max_{x_1, x_2 \in K} ||x_1 - x_2||$
- $\rho_K$ diameter of largest ball that can be inscribed into $K$
- $\sigma_K = \frac{h_K}{\rho_K}$: local shape regularity

**Lemma**

- $|\det J_K| = \frac{meas(K)}{meas(\widehat{K})}$
- $||J_K|| \leq \frac{h_K}{\rho_{\hat{K}}}$
- $||J_K^{-1}|| \leq \frac{h_{\hat{K}}}{\rho_K}$

# Local interpolation I

- For $w \in H^s(K)$ recall the $H^s$ seminorm $|w|^2_{s,K} = \sum_{|\beta|=s} ||\partial^\beta w||^2_{L^2(K)}$

**Lemma:** Let $w \in H^s(K)$ and $\widehat{w} = w \circ T_K$. There exists a constant $c$ such that

$$|\hat{w}|_{s,\hat{K}} \leq c||J_K||^s |\det J_K|^{-\frac{1}{2}} |w|_{s,K}$$
$$|w|_{s,K} \leq c||J_K^{-1}||^s |\det J_K|^{\frac{1}{2}} |\hat{w}|_{s,\hat{K}}$$

# Local interpolation II

**Theorem:** Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element with associated normed vector space $V(\widehat{K})$. Assume there exists $k$ such that

$$\mathbb{P}_K \subset \widehat{P} \subset H^{k+1}(\widehat{K}) \subset V(\widehat{K})$$

and $H^{l+1}(\widehat{K}) \subset V(\widehat{K})$ for $0 \leq l \leq k$. There exists $c > 0$ such that for all $m = 0 \ldots l+1$, $K \in \mathcal{T}_h$, $v \in H^{l+1}(K)$:

$$|v - \mathcal{I}_K^k v|_{m,K} \leq c h_K^{l+1-m} \sigma_K^m |v|_{l+1,K}$$

# Local interpolation: special cases for Lagrange finite elements

- $k = 1, l = 1, m = 0$:

$$|v - \mathcal{I}_K^k v|_{0,K} \leq ch_K^2 |v|_{2,K}$$

- $k = 1, l = 1, m = 1$:

$$|v - \mathcal{I}_K^k v|_{1,K} \leq ch_K \sigma_K |v|_{2,K}$$

# Shape regularity

- Now we discuss a family of meshes $\mathcal{T}_h$ for $h \to 0$. We want to estimate global interpolation errors and see how they possibly diminuish
- For given $\mathcal{T}_h$, assume that $h = \max_{K \in \mathcal{T}_h} h_j$
- A family of meshes is called *shape regular* if

$$\forall h, \forall K \in \mathcal{T}_h, \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0$$

- In 1D, $\sigma_K = 1$
- In 2D, $\sigma_K \leq \frac{2}{\sin \theta_K}$ where $\theta_K$ is the smallest angle

# Global interpolation error estimate

**Theorem** Let $\Omega$ be polyhedral, and let $\mathcal{T}_h$ be a shape regular family of affine meshes. Then there exists $c$ such that for all $h$, $v \in H^{l+1}(\Omega)$,

$$||v - \mathcal{I}_h^k v||_{L^2(\Omega)} + \sum_{m=1}^{l+1} h^m \left( \sum_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^k v|_{m,K}^2 \right)^{\frac{1}{2}} \leq c h^{l+1} |v|_{l+1,\Omega}$$

and

$$\lim_{h \to 0} \left( \inf_{v_h \in V_h^k} ||v - v_h||_{L^2(\Omega)} \right) = 0$$

# Global interpolation error estimate for Lagrangian finite elements, $k = 1$

- Assume $v \in H^2(\Omega)$

$$
\begin{aligned}
||v - \mathcal{I}_h^k v||_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} &\leq ch^2|v|_{2,\Omega} \\
|v - \mathcal{I}_h^k v|_{1,\Omega} &\leq ch|v|_{2,\Omega}
\end{aligned}
$$

$$
\lim_{h\to 0}\left(\inf_{v_h \in V_h^k} |v - v_h|_{1,\Omega}\right) = 0
$$

- If $v \in H^2(\Omega)$ cannot be guaranteed, estimates become worse. Example: L-shaped domain.
- These results immediately can be applied in Cea's lemma.

## Error estimates for homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, dx = \int_\Omega fv \, dx \ \forall v \in H_0^1(\Omega)$$

- Then, $\lim_{h\to 0} ||u - u_h||_{1,\Omega} = 0$.
- If $u \in H^2(\Omega)$ (e.g. convex domain, smooth coefficients), then

$$\begin{aligned} ||u - u_h||_{1,\Omega} &\leq ch|u|_{2,\Omega} \leq c'h|f|_{0,\Omega} \\ ||u - u_h||_{0,\Omega} &\leq ch^2|u|_{2,\Omega} \leq c'h^2|f|_{0,\Omega} \end{aligned}$$

and ("Aubin-Nitsche-Lemma")

$$||u - u_h||_{0,\Omega} \leq ch|u|_{1,\Omega}$$

# $H^2$-Regularity

- $u \in H^2(\Omega)$ may be *not* fulfilled e.g.
  - if $\Omega$ has re-entrant corners
  - if on a smooth part of the domain, the boundary condition type changes
  - if problem coefficients ($\lambda$) are discontinuos
- Situations differ as well between two and three space dimensions
- Delicate theory, ongoing research in functional analysis
- Consequence for simuations
  - Deterioration of convergence ratw
  - Remedy: local refinement of the discretization mesh
    - using a priori information
    - using a posteriori error estimators + automatic refinement of discretizatiom mesh

# Higher regularity

- If $u \in H^s(\Omega)$ for $s > 2$, convergence order estimates become even better for $P^k$ finite elements of order $k > 1$.
- Depending on the regularity of the solution the combination of grid adaptation and higher oder ansatz functions may be successful

# More complicated integrals

- Assume non-constant right hand side $f$, space dependent heat conduction coefficient $\kappa$.
- Right hand side integrals

$$f_i = \int_K f(x)\lambda_i(x)\ dx$$

- $P^1$ stiffness matrix elements

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ dx$$

- $P^k$ stiffness matrix elements created from higher order ansatz functions

# Quadrature rules

- *Quadrature rule*:

$$\int_K g(x)\ dx \approx |K| \sum_{l=1}^{l_q} \omega_l g(\xi_l)$$

- $\xi_l$: *nodes*, *Gauss points*
- $\omega_l$: *weights*
- The largest number $k$ such that the quadrature is exact for polynomials of order $k$ is called *order* $k_q$ of the quadrature rule, i.e.

$$\forall k \leq k_q, \forall p \in \mathbb{P}^k \int_K p(x)\ dx = |K| \sum_{l=1}^{l_q} \omega_l p(\xi_l)$$

- Error estimate:

$$\forall \phi \in \mathcal{C}^{k_q+1}(K), \left| \frac{1}{|K|} \int_K \phi(x)\ dx - \sum_{l=1}^{l_q} \omega_l g(\xi_l) \right| \leq c h_K^{k_q+1} \sup_{x \in K, |\alpha| = k_q+1} |\partial^\alpha \phi(x)|$$

# Some common quadrature rules

Nodes are characterized by the barycentric coordinates

| $d$ | $k_q$ | $l_q$ | Nodes | Weights |
|---|---|---|---|---|
| 1 | 1 | 1 | $(\frac{1}{2}, \frac{1}{2})$ | 1 |
| | 1 | 2 | $(1, 0), (0, 1)$ | $\frac{1}{2}, \frac{1}{2}$ |
| | 3 | 2 | $(\frac{1}{2} + \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6}), (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$ | $\frac{1}{2}, \frac{1}{2}$ |
| | 5 | 3 | $(\frac{1}{2}, ), (\frac{1}{2} + \sqrt{\frac{3}{20}}, \frac{1}{2} - \sqrt{\frac{3}{20}}), (\frac{1}{2} - \sqrt{\frac{3}{20}}, \frac{1}{2} + \sqrt{\frac{3}{20}})$ | $\frac{8}{18}, \frac{5}{18}, \frac{5}{18}$ |
| 2 | 1 | 1 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | 1 |
| | 1 | 3 | $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ | $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ |
| | 2 | 3 | $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})$ | $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ |
| | 3 | 4 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}), (\frac{1}{5}, \frac{3}{5}, \frac{1}{5}), (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}),$ | $-\frac{9}{16}, \frac{25}{48}, \frac{25}{48}, \frac{25}{48}$ |
| 3 | 1 | 1 | $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ | 1 |
| | 1 | 4 | $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ | $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ |
| | 2 | 4 | $(\frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5+3\sqrt{5}}{20}) \ldots$ | $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ |

# Matching of approximation order and quadrature order

- "Variational crime": instead of

$$a(u_h, v_h) = f(v_h)\ \forall v_h \in V_h$$

  we solve

$$a_h(u_h, v_h) = f_h(v_h)\ \forall v_h \in V_h$$

  where $a_h, f_h$ are derived from their exact counterparts by quadrature
- For $P^1$ finite elements, zero order quadrature for volume integrals and first order quadrature for surface intergals is sufficient to keep the convergence order estimates stated before
- The rule of thumb for the volume quadrature is that the highest order terms must be evaluated exactly if the coefficients of the PDE are constant.

# Practical realization of integrals

- Integral over barycentric coordinate function

$$\int_K \lambda_i(x)\ dx = \frac{1}{3}|K|$$

- Right hand side integrals. Assume $f(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$f_i = \int_K f(x)\lambda_i(x)\ dx \approx \frac{1}{3}|K|f_(a_i)$$

- Integral over space dependent heat conduction coefficient: Assume $\kappa(x)$ is given as a piecewise linear function with given values in the nodes of the triangulation

$$a_{ij} = \int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ dx = \frac{1}{3}(\kappa(a_0)+\kappa(a_1)+\kappa(a_2))\int_K \kappa(x)\ \nabla\lambda_i\ \nabla\lambda_j\ dx$$

# Practical realization of boundary conditions

- Robin boundary value problem

$$\begin{aligned} -\nabla \cdot \kappa \nabla u &= f \quad \text{in } \Omega \\ \kappa \nabla u + \alpha(u - g) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Weak formulation: search $u \in H^1(\Omega)$ such that

$$\int_\Omega \kappa \nabla u \nabla v \, dx + \int_{\partial\Omega} \alpha uv \, ds = \int_\Omega fv \, dx + \int_{\partial\Omega} \alpha gv \, ds \; \forall v \in H^1(\Omega)$$

- In 2D, for $P^1$ FEM, boundary integrals can be calculated by trapezoidal rule without sacrificing approximation order