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Finite element computations

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Lecture 20

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Recap

The Galerkin method

- Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive with coercivity constant α, and continuity constant γ.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v) \; \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

Global degrees of freedom

- Let $\{a_1 \dots a_N\} = \bigcup_{K \in \mathcal{T}_h} \{a_{K,1} \dots a_{K,s}\}$
- Degree of freedom map

$$
\begin{aligned}
j_{dof} : \mathcal{T}_h \times \{1 \dots s\} &\to \{1 \dots N\} \\
(K, m) &\mapsto j_{dof}(K, m) \text{ the global degree of freedom number}
\end{aligned}
$$

- Global shape functions $\phi_1, \dots, \phi_N \in W_h$ defined by

$$
\phi_i|_K(a_{K,m}) = \begin{cases} \delta_{mn} & \text{if } \exists n \in \{1 \dots s\} : j_{dof}(K, n) = i \\ 0 & \text{otherwise} \end{cases}
$$

- Global degrees of freedom $\gamma_1, \dots, \gamma_N : V_h \to \mathbb{R}$ defined by

$$
\gamma_i(v_h) = v_h(a_i)
$$

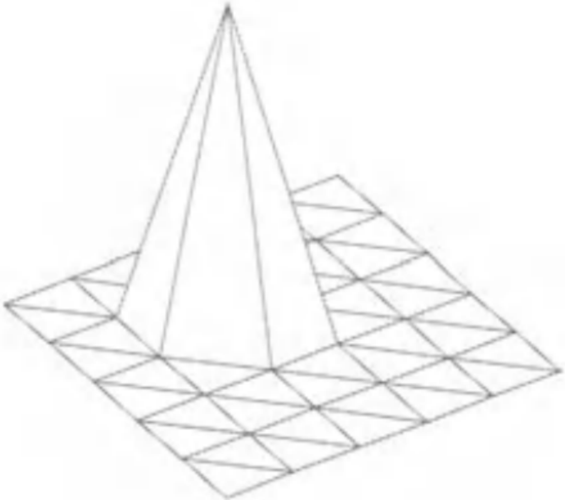
Lagrange finite element basis

- $\{\phi_1, \ldots, \phi_N\}$ is a basis of V_h, and $\gamma_1 \ldots \gamma_N$ is a basis of $\mathcal{L}(V_h, \mathbb{R})$.

Proof:

- $\{\phi_1, \ldots, \phi_N\}$ are linearly independent: if $\sum_{j=1}^{N} \alpha_j \phi_j = 0$ then evaluation at $a_1 \ldots a_N$ yields that $\alpha_1 \ldots \alpha_N = 0$.
- Let $v_h \in V_h$. It is single valued in $a_1 \ldots a_N$. Let $w_h = \sum_{j=1}^{N} v_h(a_j)\phi_j$. Then for all $K \in \mathcal{T}_h$, $v_h|_K$ and $w_h|_K$ coincide in the local nodes $a_{K,1} \ldots a_{K,2}$, and by unisolvence, $v_h|_K = w_h|_K$.

P^1 global shape functions



From the Galerkin method to the matrix equation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of V_h.
- Then, we have the representation $u_h = \sum_{j=1}^{n} u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

it is actually sufficient to require

$$\begin{aligned} a(u_h, \phi_i) &= f(\phi_i) \; (i = 1 \dots n) \\ a\left(\sum_{j=1}^{n} u_j \phi_j, \phi_i\right) &= f(\phi_i) \; (i = 1 \dots n) \\ \sum_{j=1}^{n} a(\phi_j, \phi_i) u_j &= f(\phi_i) \; (i = 1 \dots n) \\ AU &= F \end{aligned}$$

with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j),\ F = (f_i), f_i = F(\phi_i),\ U = (u_i)$.
- Matrix dimension is $n \times n$.

Stiffness matrix calculation for Laplace operator for P1 FEM

$$
\begin{aligned}
a_{ij} = a(\phi_i, \phi_j) &= \int_\Omega \nabla \phi_i \nabla \phi_j \; dx \\
&= \int_\Omega \sum_{K \in \mathcal{T}_h} \nabla \phi_i|_K \nabla \phi_j|_K \; dx
\end{aligned}
$$

Assembly loop:
Set $a_{ij} = 0$.
For each $K \in \mathcal{T}_h$:
For each $m, n = 0 \dots d$:

$$
\begin{aligned}
s_{mn} &= \nabla \lambda_m \nabla \lambda_n \; dx \\
a_{j_{dof}(K,m), j_{dof}(K,n)} &= a_{j_{dof}(K,m), j_{dof}(K,n)} + s_{mn}
\end{aligned}
$$

Local stiffness matrix calculation for P1 FEM

$a_0 \dots a_d$: vertices of the simplex K, $a \in K$.

Barycentric coordinates: $\lambda_j(a) = \frac{|K_j(a)|}{|K|}$

For indexing modulo d+1 we can write

$$|K| = \frac{1}{d!} \det\left(a_{j+1} - a_j, \dots a_{j+d} - a_j\right)$$
$$|K_j(a)| = \frac{1}{d!} \det\left(a_{j+1} - a, \dots a_{j+d} - a\right)$$

From this information, we can calculate $\nabla \lambda_j(x)$ (which are constant vectors due to linearity) and the corresponding entries of the local stiffness matrix

$$s_{ij} = \int_K \nabla \lambda_i \nabla \lambda_j \, dx$$

Local stiffness matrix calculation for P1 FEM in 2D

$a_0 = (x_0, y_0) \dots a_d = (x_2, y_2)$: vertices of the simplex K, $a = (x, y) \in K$.

Barycentric coordinates: $\lambda_j(x, y) = \frac{|K_j(x,y)|}{|K|}$

For indexing modulo d+1 we can write

$$|K| = \frac{1}{2} \det \begin{pmatrix} x_{j+1} - x_j & x_{j+2} - x_j \\ y_{j+1} - y_j & y_{j+2} - y_j \end{pmatrix}$$
$$|K_j(x, y)| = \frac{1}{2} \det \begin{pmatrix} x_{j+1} - x & x_{j+2} - x \\ y_{j+1} - y & y_{j+2} - y \end{pmatrix}$$

Therefore, we have

$$|K_j(x, y)| = \frac{1}{2} \left((x_{j+1} - x)(y_{j+2} - y) - (x_{j+2} - x)(y_{j+1} - y) \right)$$
$$\partial_x |K_j(x, y)| = \frac{1}{2} \left((y_{j+1} - y) - (y_{j+2} - y) \right) = \frac{1}{2} (y_{j+1} - y_{j+2})$$
$$\partial_y |K_j(x, y)| = \frac{1}{2} \left((x_{j+2} - x) - (x_{j+1} - x) \right) = \frac{1}{2} (x_{j+2} - x_{j+1})$$

Local stiffness matrix calculation for P1 FEM in 2D II

$$s_{ij} = \int_K \nabla\lambda_i \nabla\lambda_j \; dx = \frac{|K|}{4|K|^2} \left(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}\right) \begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix}$$

So, let $V = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix}$

Then

$$\begin{aligned} x_1 - x_2 &= V_{00} - V_{01} \\ y_1 - y_2 &= V_{10} - V_{11} \end{aligned}$$

and

$$\begin{aligned} 2|K| \; \nabla\lambda_0 &= \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} V_{10} - V_{11} \\ V_{01} - V_{00} \end{pmatrix} \\ 2|K| \; \nabla\lambda_1 &= \begin{pmatrix} y_2 - y_0 \\ x_0 - x_2 \end{pmatrix} = \begin{pmatrix} V_{11} \\ -V_{01} \end{pmatrix} \\ 2|K| \; \nabla\lambda_2 &= \begin{pmatrix} y_0 - y_1 \\ x_1 - x_0 \end{pmatrix} = \begin{pmatrix} -V_{10} \\ V_{00} \end{pmatrix} \end{aligned}$$

Degree of freedom map representation for P1 finite elements

- List of global nodes $a_0 \dots a_N$: two dimensional array of coordinate values with N rows and d columns
- Local-global degree of freedom map: two-dimensional array C of index values with N_{el} rows and $d+1$ columns such that $C(i,m) = j_{dof}(K_i, m)$.
- The mesh generator triangle generates this information directly