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Finite element construction

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Lecture 18

Jürgen Fuhrmann

juergen.fuhrmann@wias-berlin.de

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Recap (Finite Elements)

Heat conduction revisited: Derivation of weak formulation

+- Sobolev space theory provides the necessary framework to formulate existence and uniqueness of solutions of PDEs.

- Heat conduction equation with homogeneous Dirichlet boundary conditions:

$$
\begin{aligned}
-\nabla \cdot \lambda \nabla u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

Multiply and integrate with an arbitrary *test function* from $C_0^\infty(\Omega)$:

$$
\begin{aligned}
-\int_\Omega \nabla \cdot \lambda \nabla u v \, dx &= \int_\Omega f v \, dx \\
\int_\Omega \lambda \nabla u \nabla v \, dx &= \int_\Omega f v \, dx
\end{aligned}
$$

Weak formulation of homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, dx = \int_\Omega fv \, dx \ \forall v \in H_0^1(\Omega)$$

- Then,

$$a(u, v) := \int_\Omega \lambda \nabla u \nabla v \, dx$$

  is a self-adjoint bilinear form defined on the Hilbert space $H_0^1(\Omega)$.

It is bounded due to Cauchy-Schwarz:

$$|a(u, v)| = |\lambda| |\int_\Omega \nabla u \nabla v \, dx| \leq ||u||_{H_0^1(\Omega)} \cdot ||v||_{H_0^1(\Omega)}$$

- $f(v) = \int_\Omega fv \, dx$ is a linear functional on $H_0^1(\Omega)$. For Hilbert spaces V the dual space V' (the space of linear functionals) can be identified with the space itself.

The Lax-Milgram lemma

Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive, i.e.

$$\exists \alpha > 0 : \forall u \in V, a(u,u) \geq \alpha ||u||_V^2.$$

Then the problem: find $u \in V$ such that

$$a(u,v) = f(v) \; \forall v \in V$$

admits one and only one solution with an a priori estimate

$$||u||_V \leq \frac{1}{\alpha} ||f||_{V'}$$

Heat conduction revisited

Let $\lambda > 0$. Then the weak formulation of the heat conduction problem: search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, dx = \int_\Omega fv \, dx \; \forall v \in H_0^1(\Omega)$$

has an unique solution.

Proof: $a(u, v)$ is cocercive:

$$a(u, v) = \int_\Omega \lambda \nabla u \nabla u \, dx = \lambda ||u||^2_{H_0^1(\Omega)}$$

$\square$

Weak formulation of inhomogeneous Dirichlet problem

$$
\begin{aligned}
-\nabla \cdot \lambda \nabla u &= f \text{ in } \Omega \\
u &= g \text{ on } \partial\Omega
\end{aligned}
$$

If g is smooth enough, there exists a *lifting* $u_g \in H^1(\Omega)$ such that $u_g|_{\partial\Omega} = g$. Then, we can re-formulate:

$$
\begin{aligned}
-\nabla \cdot \lambda \nabla (u - u_g) &= f + \nabla \cdot \lambda \nabla u_g \text{ in } \Omega \\
u - u_g &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

- Search $u \in H^1(\Omega)$ such that

$$
\begin{aligned}
u &= u_g + \phi \\
\int_\Omega \lambda \nabla \phi \nabla v \, dx &= \int_\Omega f v \, dx + \int_\Omega \lambda \nabla u_g \nabla v \quad \forall v \in H^1_0(\Omega)
\end{aligned}
$$

Here, necessarily, $\phi \in H^1_0(\Omega)$ and we can apply the theory for the homogeneous Dirichlet problem.

Weak formulation of Robin problem

$$-\nabla \cdot \lambda \nabla u = f \text{ in } \Omega$$
$$\lambda \nabla u \cdot \mathbf{n} + \alpha(u - g) = 0 \text{ on } \partial\Omega$$

Multiply and integrate with an arbitrary *test function* from $C_c^\infty(\Omega)$:

$$-\int_\Omega (\nabla \cdot \lambda \nabla u) v \, dx = \int_\Omega fv \, dx$$
$$\int_\Omega \lambda \nabla u \nabla v \, dx + \int_{\partial\Omega} (\lambda \nabla u \cdot \mathbf{n}) v ds = \int_\Omega fv \, dx$$
$$\int_\Omega \lambda \nabla u \nabla v \, dx + \int_{\partial\Omega} \alpha uv \, ds = \int_\Omega fv \, dx + \int_{\partial\Omega} \alpha gv \, ds$$

Weak formulation of Robin problem II

- Let

$$a^R(u,v) := \int_\Omega \lambda \nabla u \nabla v \, dx + \int_{\partial\Omega} \alpha uv \, ds$$
$$f^R(v) := \int_\Omega fv \, dx + \int_{\partial\Omega} \alpha gv \, ds$$

  The integrals over $\partial\Omega$ must be understood in the sense of the trace space $H^{\frac{1}{2}}(\partial\Omega)$.

- Search $u \in H^1(\Omega)$ such that

$$a^R(u,v) = f^R(v) \; \forall v \in H^1(\Omega)$$

- If $\lambda > 0$ and $\alpha > 0$ then $a^R(u,v)$ is cocercive.

Neumann boundary conditions

Homogeneous Neumann:

$$\lambda \nabla u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega$$

Inhomogeneous Neumann:

$$\lambda \nabla u \cdot \mathbf{n} = g \text{ on } \partial\Omega$$

Weak formulation:

- Search $u \in H^1(\Omega)$ such that

$$\int_\omega \nabla u \nabla v dx = \int_{\partial\Omega} gvds \; \forall v \in H^1(\Omega)$$

Not coercive due to the fact that we can add an arbitrary constant to u and $a(u, u)$ stays the same!

Further discussion on boundary conditions

- Mixed boundary conditions:
  One can have differerent boundary conditions on different parts of the boundary. In particular, if Dirichlet or Robin boundary conditions are applied on at least a part of the boundary of measure larger than zero, the binlinear form becomes coercive.
- Natural boundary conditions: Robin, Neumann
  These are imposed in a "natural" way in the weak formulation
- Essential boundary conditions: Dirichlet
  Explicitely imposed on the function space
- Coefficients $\lambda, \alpha \ldots$ can be functions.

The Dirichlet penalty method

- Robin problem: search $u_\alpha \in H^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u_\alpha \nabla v \, dx + \int_{\partial\Omega} \alpha u_\alpha v \, ds = \int_\Omega fv \, dx + \int_{\partial\Omega} \alpha g v \, ds \forall v \in H^1(\Omega)$$

- Dirichlet problem: search $u \in H^1(\Omega)$ such that

$$u = u_g + \phi \quad \text{where } u_g|_{\partial\Omega} = g$$

$$\int_\Omega \lambda \nabla \phi \nabla v \, dx = \int_\Omega fv \, dx + \int_\Omega \lambda \nabla u_g \nabla v \;\; \forall v \in H^1_0(\Omega)$$

- Penalty limit:

$$\lim_{\alpha \to \infty} u_\alpha = u$$

- Formally, the convergence rate is quite low
- Implementing Dirichlet boundary conditions directly leads to a number of technical problems
- Implementing the penalty method is technically much simpler
- Proper way of handling the parameter leads to exact fulfillment of Dirichlet boundary condition in the floating point precision

The Galerkin method I

- Weak formulations “live” in Hilbert spaces which essentially are infinite dimensional
- For computer representations we need finite dimensional approximations
- The finite volume method provides one possible framework which in many cases is close to physical intuition. However, its error analysis is hard.
- The Galerkin method and its modifications provide a general scheme for the derivation of finite dimensional appoximations

The Galerkin method II

- Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive with coercivity constant α, and continuity constant γ.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v) \; \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

Céa's lemma

- What is the connection between u and u_h ?
- Let $v_h \in V_h$ be arbitrary. Then

$$\begin{aligned}
\alpha ||u - u_h||^2 &\leq a(u - u_h, u - u_h) \quad \text{(Coercivity)} \\
&= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
&= a(u - u_h, u - v_h) \quad \text{(Galerkin Orthogonality)} \\
&\leq \gamma ||u - u_h|| \cdot ||u - v_h|| \quad \text{(Boundedness)}
\end{aligned}$$

- As a result

$$||u - u_h|| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u - v_h||$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace V_h.

From the Galerkin method to the matrix equation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of V_h.
- Then, we have the representation $u_h = \sum_{j=1}^{n} u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \; \forall v_h \in V_h$$

  it is actually sufficient to require

$$\begin{aligned} a(u_h, \phi_i) &= f(\phi_i) \; (i = 1 \dots n) \\ a\left(\sum_{j=1}^{n} u_j \phi_j, \phi_i\right) &= f(\phi_i) \; (i = 1 \dots n) \\ \sum_{j=1}^{n} a(\phi_j, \phi_i) u_j &= f(\phi_i) \; (i = 1 \dots n) \\ AU &= F \end{aligned}$$

  with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j)$, $F = (f_i), f_i = F(\phi_i)$, $U = (u_i)$.
- Matrix dimension is $n \times n$. Matrix sparsity ?

Obtaining a finite dimensional subspace

- Let $\Omega = (a, b) \subset \mathbb{R}^1$
- Let $a(u, v) = \int_\Omega \lambda(x) \nabla u \nabla v dx$.
- Analysis I provides a finite dimensional subspace: the space of sin/cos functions up to a certain frequency $\Rightarrow$ *spectral method*
- Ansatz functions have global support $\Rightarrow$ full $n \times n$ matrix
- OTOH: rather fast convergence for smooth data
- Generalization to higher dimensions possible
- Big problem in irregular domains: we need the eigenfunction basis of some operator. . .
- Spectral methods are successful in cases where one has regular geometry structures and smooth/constant coefficients – e.g. "Spectral Einstein Code"

The finite element idea

- Choose basis functions with local support. In this case, the matrix becomes sparse, as only integrals of basis function pairs with overlapping support contribute to the matrix.
- Linear finite elements in $\Omega = (a, b) \subset \mathbb{R}^1$:
- Partition $a = x_1 \leq x_2 \leq \cdots \leq x_n = b$
- Basis functions (for $i = 1 \ldots n$)

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & i > 1, x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & i < n, x \in (x_i, x_{i+1}) \\ 0, & \text{else} \end{cases}$$

- Any function $u_h \in V_h = \mathrm{span}\{\phi_1 \ldots \phi_n\}$ is piecewise linear, and the coefficients in the representation $u_h = \sum_{i=1}^n u_i \phi_i$ are the values $u_h(x_i)$.
- Fortunately, we are working with a weak formulation, and weak derivatives are well defined !

1D matrix elements

$(\lambda = 1,\ x_{i+1} - x_i = h)$ - The integrals are nonzero for $i = j, i+1 = j, i-1 = j$
Let $j = i + 1$

$$a_{ij} = a(\phi_i, \phi_{i+1}) = \int_\Omega \nabla\phi_i \nabla\phi_j dx = \int_{x_i}^{x_{i+1}} \nabla\phi_i \nabla\phi_j dx = -\int_{x_i}^{x_{i+1}} \frac{1}{h^2} dx$$
$$= \frac{1}{h} dx$$

Similarly, $a(\phi_i, \phi_{i-1}) = -\frac{1}{h}$

For $1 < i < N$:

$$a_{ii} = a(\phi_i, \phi_i) = \int_\Omega \nabla\phi_i \nabla\phi_i dx = \int_{x_{i-1}}^{x_{i+1}} \nabla\phi_i \nabla\phi_i dx = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} dx$$
$$= \frac{2}{h} dx$$

For $i = 1$ or $i = N$, $a(\phi_i, \phi_i) = \frac{1}{h}$

1D matrix elements II

Adding the boundary integrals yields

$$A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix}$$

... the same matrix as for the finite volume method...

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Finite Elements in higher dimensions

(after Ern/Guermond)

Definition of a Finite Element (Ciarlet)

Triplet $\{K, P, \Sigma\}$ where

- $K \subset \mathbb{R}^d$: compact, connected Lipschitz domain with non-empty interior
- P: finite dimensional vector space of functions $p : K \to \mathbb{R}^m$ (mostly, $m = 1, m = d$)
- $\Sigma = \{\sigma_1 \dots \sigma_s\} \subset \mathcal{L}(P, \mathbb{R})$: set of linear forms defined on P called *local degrees of freedom* such that the mapping

$$\begin{aligned} \Lambda_\Sigma : P &\to \mathbb{R}^s \\ p &\mapsto (\sigma_1(p) \dots \sigma_s(p)) \end{aligned}$$

is bijective, i.e. Σ is a basis of $\mathcal{L}(P, \mathbb{R})$.

Local shape functions

- Due to bijectivity of Λ_Σ, for any finite element $\{K, P, \Sigma\}$, there exists a basis $\{\theta_1 \dots \theta_s\} \subset P$ such that

$$\sigma_i(\theta_j) = \delta_{ij} \quad (1 \le i, j \le s)$$

- Elements of such a basis are called *local shape functions*

Unisolvence

- Bijectivity of Λ_Σ is equivalent to the condition

$$\forall(\alpha_1 \dots \alpha_s) \in \mathbb{R}^s \; \exists! p \in P \text{ such that } \sigma_i(p) = \alpha_i \quad (1 \leq i \leq s)$$

  i.e. for any given tuple of values $a = (\alpha_1 \dots \alpha_s)$ there is a unique polynomial $p \in P$ such that $\Lambda_\Sigma(p) = a$.
- Equivalent to *unisolvence*:

$$\begin{cases} \dim P = |\Sigma| = s \\ \forall p \in P : \; \sigma_i(p) = 0 \, (i = 1 \dots s) \; \Rightarrow \; p = 0 \end{cases}$$

Lagrange finite elements

- A finite element $\{K, P, \Sigma\}$ is called *Lagrange* finite element (or *nodal* finite element) if there exist a set of points $\{a_1 \dots a_s\} \subset K$ such that

$$\sigma_i(p) = p(a_i) \quad 1 \leq i \leq s$$

- $\{a_1 \dots a_s\}$: *nodes* of the finite element
- *nodal basis: $\{\theta_1 \dots \theta_s\} \subset P$ such that

$$\theta_j(a_i) = \delta_{ij} \quad (1 \leq i, j \leq s)$$

Hermite finite elements

- All or a part of degrees of freedoms defined by derivatives of p in some points

Local interpolation operator

- Let $\{K, P, \Sigma\}$ be a finite element with shape function bases $\{\theta_1 \dots \theta_s$. Let $V(K)$ be a normed vector space of functions $v : K \to \mathbb{R}^m$ such that
  - $P \subset V(K)$
  - The linear forms in Σ can be extended to be defined on $V(K)$
- *local interpolation operator*

$$\begin{aligned} \mathcal{I}_K : V(K) &\to P \\ v &\mapsto \sum_{i=1}^{s} \sigma_i(v)\theta_i \end{aligned}$$

- P is invariant under the action of $\mathcal{I}_K$, i.e. $\forall p \in P, \mathcal{I}_K(p) = p$:
  - Let $p = \sum_{j=1}^{s} \alpha_j \theta_j$ Then,

$$\begin{aligned} \mathcal{I}_K(p) = \sum_{i=1}^{s} \sigma_i(p)\theta_i &= \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_j \sigma_i(\theta_j)\theta_i \\ &= \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_j \delta_{ij} \theta_i = \sum_{j=1}^{s} \alpha_j \theta_j \end{aligned}$$

Local Lagrange interpolation operator

- Let $V(K) = (\mathcal{C}^0(K))^m$

$$\begin{aligned} \mathcal{I}_K : V(K) &\to P \\ v \mapsto I_K v &= \sum_{i=1}^{s} v(a_i)\theta_i \end{aligned}$$

Simplices

- Let $\{a_0 \dots a_d\} \subset \mathbb{R}^d$ such that the d vectors $a_1 - a_0 \dots a_d - a_0$ are linearly independent. Then the convex hull K of $a_0 \dots a_d$ is called *simplex*, and $a_0 \dots a_d$ are called *vertices* of the simplex.
- *Unit simplex*: $a_0 = (0...0), a_1 = (0, 1 \dots 0) \dots a_d = (0 \dots 0, 1)$.

$$K = \left\{ x \in \mathbb{R}^d : x_i \geq 0 \; (i = 1 \dots d) \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}$$

- A general simplex can be defined as an image of the unit simplex under some affine transformation
- F_i: face of K opposite to a_i
- $\mathbf{n}_i$: outward normal to F_i

Barycentric coordinates

- Let K be a simplex.
- Functions λ_i ($i = 0 \dots d$):

$$\begin{aligned} \lambda_i : \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto \lambda_i(x) = 1 - \frac{(x - a_i)\cdot \mathbf{n}_i}{(a_j - a_i)\cdot \mathbf{n}_i} \end{aligned}$$

  where a_j is any vertex of K situated in F_i.
- For $x \in K$, one has

$$\begin{aligned} 1 - \frac{(x - a_i)\cdot \mathbf{n}_i}{(a_j - a_i)\cdot \mathbf{n}_i} &= \frac{(a_j - a_i)\cdot \mathbf{n}_i - (x - a_i)\cdot \mathbf{n}_i}{(a_j - a_i)\cdot \mathbf{n}_i} \\ &= \frac{(a_j - x)\cdot \mathbf{n}_i}{(a_j - a_i)\cdot \mathbf{n}_i} = \frac{\mathrm{dist}(x, F_i)}{\mathrm{dist}(a_i, F_i)} \\ &= \frac{\mathrm{dist}(x, F_i)|F_i|/d}{\mathrm{dist}(a_i, F_i)|F_i|/d} \\ &= \frac{\mathrm{dist}(x, F_i)|F_i|}{|K|} \end{aligned}$$

  i.e. $\lambda_i(x)$ is the ratio of the volume of the simplex $K_i(x)$ made up of x and the vertices of F_i to the volume of K.

Barycentric coordinates II

- $\lambda_i(a_j) = \delta_{ij}$
- $\lambda_i(x) = 0 \; \forall x \in F_i$
- $\sum_{i=0}^{d} \lambda_i(x) = 1 \; \forall x \in \mathbb{R}^d$
  (just sum up the volumes)
- $\sum_{i=0}^{d} \lambda_i(x)(x - a_i) = 0 \; \forall x \in \mathbb{R}^d$
  (due to $\sum \lambda_i(x)x = x$ and $\sum \lambda_i a_i = x$ as the vector of linear coordinate functions)
- Unit simplex:
  - $\lambda_0(x) = 1 - \sum_{i=1}^{d} x_i$
  - $\lambda_i(x) = x_i$ for $1 \leq i \leq d$

Polynomial space $\mathbb{P}_k$

- Space of polynomials in $x_1 \dots x_d$ of total degree $\le k$ with real coefficients $\alpha_{i_1 \dots i_d}$:

$$\mathbb{P}_k = \left\{ p(x) = \sum_{\substack{0 \le i_1 \dots i_d \le k \\ i_1 + \dots + i_d \le k}} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} \right\}$$

- Dimension:

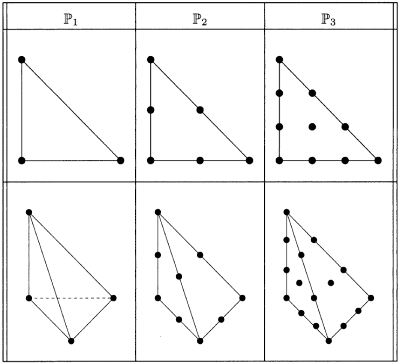
$$\dim \mathbb{P}_k = \binom{d+k}{k} = \begin{cases} k+1, & d = 1 \\ \frac{1}{2}(k+1)(k+2), & d = 2 \\ \frac{1}{6}(k+1)(k+2)(k+3), & d = 3 \end{cases}$$

$$\dim \mathbb{P}_1 = d + 1$$

$$\dim \mathbb{P}_2 = \begin{cases} 3, & d = 1 \\ 6, & d = 2 \\ 10, & d = 3 \end{cases}$$

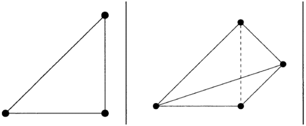
$\mathbb{P}_k$ simplex finite elements

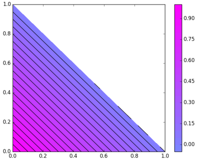
- K: simplex spanned by $a_0 \dots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_k$, such that $s = \dim P_k$
- For $0 \leq i_0 \dots i_d \leq k$, $i_0 + \dots + i_d = k$, let the set of nodes be defined by the points $a_{i_1 \dots i_d;k}$ with barycentric coordinates $(\frac{i_0}{k} \dots \frac{i_d}{k})$.
  Define Σ by $\sigma_{i_1 \dots i_d;k}(p) = p(a_{i_1 \dots i_d;k})$.

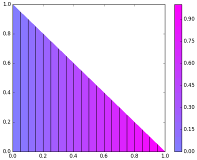


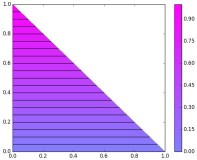
$\mathbb{P}_1$ simplex finite elements

- K: simplex spanned by $a_0 \ldots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_1$, such that $s = d + 1$
- Nodes $\equiv$ vertices
- Basis functions $\equiv$ barycentric coordinates





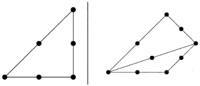


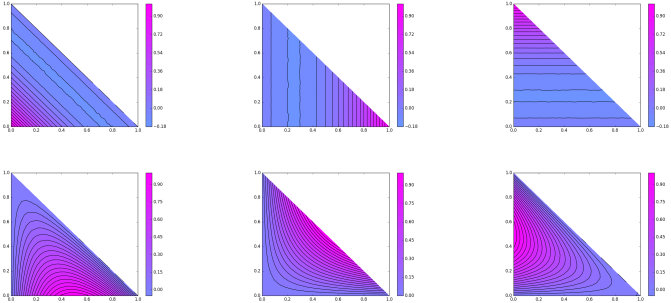


$\mathbb{P}_2$ simplex finite elements

- K: simplex spanned by $a_0 \dots a_d$ in $\mathbb{R}^d$
- $P = \mathbb{P}_2$, Nodes $\equiv$ vertices + edge midpoints
- Basis functions:

$$\lambda_i(2\lambda_i - 1), (0 \le i \le d); \quad 4\lambda_i\lambda_j, \quad (0 \le i < j \le d) \quad (\text{"edge bubbles"})$$





Cuboids

- Given intervals $I_i = [c_i, d_i]$, $i = 1 \dots d$ such that $c_i < d_i$.
- *Cuboid*:

$$K = \prod_{i=1}^{d} [c_i, d_i]$$

- Local coordinate vector $(t_1 \dots t_d) \in [0,1]^d$
- Unique representation of $x \in K$: $x_i = c_i + t_i(d_i - c_i)$ for $i = 1 \dots d$.
- Bijective mapping $[0,1]^d \to K$.

Polynomial space $\mathbb{Q}_k$

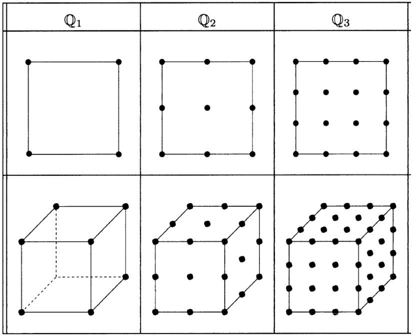
- Space of polynomials of degree at most k in each variable
- $d = 1 \Rightarrow \mathbb{Q}_k = \mathbb{P}_k$
- $d > 1$:

$$\mathbb{Q}_k = \left\{ p(x) = \sum_{0 \le i_1 \dots i_d \le k} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} \right\}$$

- $\dim \mathbb{Q}_k = (k+1)^d$

$\mathbb{Q}_k$ cuboid finite elements

- K: cuboid spanned by intervals $[c_i, d_i]$, $i = 1 \dots d$
- $P = \mathbb{Q}_k$
- For $0 \le i_0 \dots i_d \le k$, let the set of nodes be defined by the points $a_{i_1 \dots i_d;k}$ with local coordinates $(\frac{i_0}{k} \dots \frac{i_d}{k})$.
  Define Σ by $\sigma_{i_1 \dots i_d;k}(p) = p(a_{i_1 \dots i_d;k})$.



General finite elements

- Simplicial finite elements can be defined on triangulations of polygonal domains. During the course we will stick to this case.
- A curved domain Ω may be approximated by a polygonal domain Ω_h which is then triangulated. During the course, we will ignore this difference.
- As we have seen, more general elements are possible: cuboids, but and $T_m|_{\widehat{F}} = T_n|_{\widehat{F}}$also prismatic elements etc.
- Curved geometries are possible. Isoparametric finite elements use and $T_m|_{\widehat{F}} = T_n|_{\widehat{F}}$the polynomial space to define a mapping of some polyghedral reference element to an element with curved boundary

Conformal triangulations

- Let $\mathcal{T}_h$ be a subdivision of the polygonal domain $\Omega \subset \mathbb{R}^d$ into non-intersecting compact simplices K_m, $m = 1 \dots n_e$:

$$\overline{\Omega} = \bigcup_{m=1}^{n_e} K_m$$

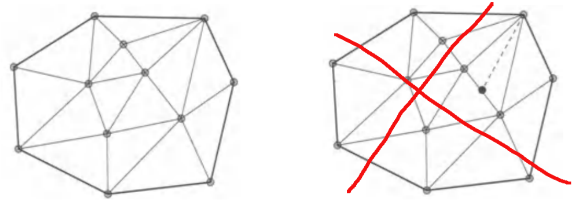
- Each simplex can be seen as the image of a affine transormation of a reference (e.g. unit) simplex $\widehat{K}$:

$$K_m = T_m(\widehat{K})$$

- We assume that it is conformal, i.e. if K_m, K_n have a $d-1$ dimensional intersection $F = K_m \cap K_n$, then there is a face $\widehat{F}$ of $\widehat{K}$ and renumberings of the vertices of K_n, K_m such that $F = T_m(\widehat{F}) = T_n(\widehat{F})$ and $T_m|_{\widehat{F}} = T_n|_{\widehat{F}}$

Conformal triangulations II

- $d = 1$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex
- $d = 2$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge



- $d = 3$: Each intersection $F = K_m \cap K_n$ is either empty or a common vertex or a common edge or a common face
- Triangulations corresponding to simplicial complexes are conformal
- Delaunay triangulations are conformal

Reference finite element

- Let $\{\widehat{P}, \widehat{K}, \widehat{\Sigma}\}$ be a fixed finite element
- Let T_K be some affine transformation and $K = T_K(\widehat{K})$
- There is a linear bijective mapping ψ_K between functions on K and functions on $\widehat{K}$:

$$\begin{aligned} \psi_K : V(K) &\to V(\widehat{K}) \\ f &\mapsto f \circ T_K \end{aligned}$$

- Let
  - $K = T_K(\widehat{K})$\$
  - $P_K = \{\psi_K^{-1}(\widehat{p}); \widehat{p} \in \widehat{P}\}$,
  - $\Sigma_K = \{\sigma_{K,i}, i = 1 \ldots s : \sigma_{K,i}(p) = \widehat{\sigma}_i(\psi_K(p))\}$ Then $\{K, P_K, \Sigma_K\}$ is a finite element.

Commutativity of interpolation and reference mapping

- $\mathcal{I}_{\hat{K}} \circ \psi_K = \psi_K \circ \mathcal{I}_K$,
  i.e. the following diagram is commutative:

$$\begin{array}{ccc} V(K) & \xrightarrow{\psi_K} & V(\widehat{K}) \\ \big\downarrow \mathcal{I}_K & & \big\downarrow \mathcal{I}_{\hat{K}} \\ P_K & \xrightarrow{\psi_K} & P_{\widehat{K}} \end{array}$$

Global interpolation operator $\mathcal{I}_h$

- Let $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ be a triangulation of Ω.
- Domain:

$$D(\mathcal{I}_h) = \{v \in (L^1(\Omega))^m \text{ such that } \forall K \in \mathcal{T}_h, v|_K \in V(K)\}$$

- For all $v \in D(\mathcal{I}_h)$, define $\mathcal{I}_h v$ via

$$\mathcal{I}_h v|_K = \mathcal{I}_K(v|_K) = \sum_{i=1}^{s} \sigma_{K,i}(v|_K)\theta_{K,i} \; \forall K \in \mathcal{T}_h,$$

Assuming $\theta_{K,i} = 0$ outside of K, one can write

$$\mathcal{I}_h v = \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{s} \sigma_{K,i}(v|_K)\theta_{K,i},$$

mapping $D(\mathcal{I}_h)$ to the *approximation space*

$$W_h = \{v_h \in (L^1(\Omega))^m \text{ such that } \forall K \in \mathcal{T}_h, v_h|_K \in P_K\}$$

H^1-Conformal approximation using Lagrangian finite elemenents

- Let V be a Banach space of functions on Ω. The approximation space W_h is said to be V-conformal if $W_h \subset V$.
- Non-conformal approximations are possible, we will stick to the conformal case.
- Conformal subspace of W_h with zero jumps at element faces:

$$V_h = \{v_h \in W_h : \forall n, m, K_m \cap K_n \neq 0 \Rightarrow (v_h|_{K_m})_{K_m \cap K_n} = (v_h|_{K_n})_{K_m \cap K_n}\}$$

- Then: $V_h \subset H^1(\Omega)$.

Happy Holidays!

Next lecture: Jan. 5, 2017