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Weak formulations and finite elements

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Lecture 16

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Recap (Delaunay, Sobolev spaces)

Delaunay triangulations

- Given a finite point set $X \subset \mathbb{R}^d$. Then there exists simplicial a complex called *Delaunay triangulation* of this point set such that
  - X is the set of vertices of the triangulation
  - The union of all its simplices is the convex hull of X.
  - (Delaunay property): For any given d-simplex $\Sigma \subset \Omega$ belonging to the triangulation, the interior of its circumsphere does not contain any vertex $x_k \in X$.
- Assume that the points of X are in general position, i.e. no $n + 2$ points lie on one sphere. Then the Delaunay triangulation is unique.

Voronoi diagram

- Given a finite point set $X \subset \mathbb{R}^d$. Then the Voronoi diagram is a partition of $\mathbb{R}^d$ into convex nonoverlapping polygonal regions defined as

$$\mathbb{R}^d = \bigcup_{k=1}^{N_x} V_k$$

$$V_k = \{x \in \mathbb{R}^d : ||x - x_k|| < ||x - x_l|| \forall x_l \in X, l \neq k\}$$

Voronoi - Delaunay duality

- Given a point set $X \subset \mathbb{R}^d$ in general position. Then its Delaunay triangulation and its Voronoi diagram are dual to each other:
  - Two Voronoi cells V_k, V_l have a common facet if and only if $\overline{x_k x_l}$ is an edge of the triangulation.

Boundary conforming Delaunay triangulations

- Domain $\Omega \subset \mathbb{R}^n$ (we will discuss only $n = 2$) with polygonal boundary $\partial\Omega$.
- Partition (triangulation) $\Omega = \bigcup_{s=1}^{N_\Sigma} \Sigma$ into non-overlapping simplices Σ_s such that this partition represents a simplicial complex. Regard the set of nodes $X = \{x_1 \dots x_{N_x}\}$.
- It induces a partition of the boundary into lower dimensional simplices: $\partial\Omega = \bigcup_{t=1}^{N_\sigma} \sigma_t$. We assume that in 3D, the set $\{\sigma_t\}_{t=1}^{N_\sigma}$ includes all edges of surface triangles as well. For any given lower ($d-1$ or $d-2$) dimensional simplex σ, its *diametrical sphere* is defined as the smallest sphere containing all its vertices.
- *Boundary conforming Delaunay property*:
  - (Delaunay property): For any given d-simplex $\Sigma_s \subset \Omega$, the interior of its circumsphere does not contain any vertex $x_k \in X$.
  - (Gabriel property) For any simplex $\sigma_t \subset \partial\Omega$, the interior of its diametrical sphere does not contain any vertex $x_k \in X$.
- Equivalent formulation in 2D:
  - For any two triangles with a common edge, the sum of their respective angles opposite to that edge is less or equal to 180°.
  - For any triangle sharing an edge with $\partial\Omega$, its angle opposite to that edge is less or equal to 90°.

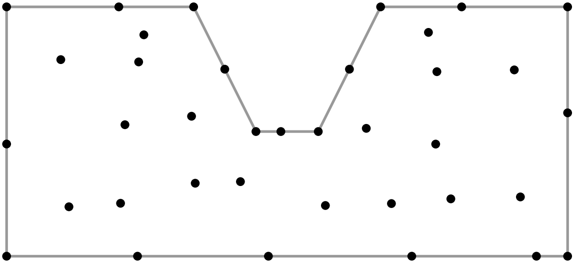
Restricted Voronoi diagram

- Given a boundary conforming Delaunay discretization of Ω, the *restricted Voronoi diagram* consists of the *restricted Voronoi cells* corresponding to the node set X defined by

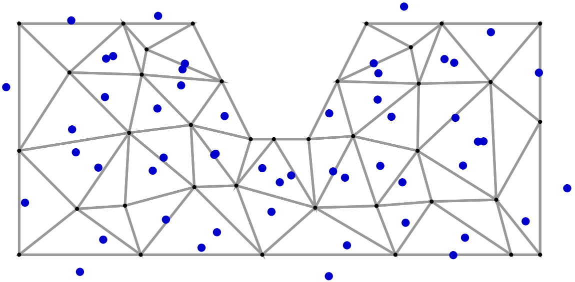
$$\omega_k = V_k \cap \Omega = \{x \in \Omega : ||x - x_k|| < ||x - x_l|| \forall x_l \in X, l \neq k\}$$

- These restricted Voronoi cells are used as control volumes in a finite volume discretization

Piecewise linear description of computational domain with given point cloud

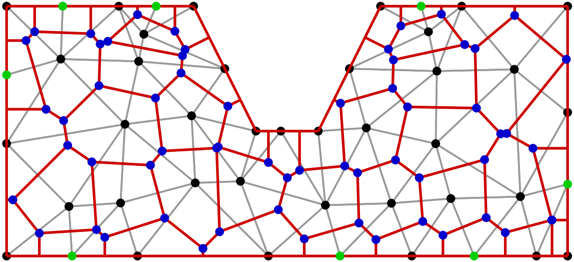


Delaunay triangulation of domain and triangle circumcenters.



- Blue: triangle circumcenters
- Some boundary triangles have larger than 90° angles opposite to the boundary $\Rightarrow$ their circumcenters are outside of the domain

Boundary conforming Delaunay triangulation



- Automatically inserted additional points at the boundary (green dots)
- Restricted Voronoi cells (red).

General approach to triangulations

- Obtain piecewise linear descriptiom of domain
- Call mesh generator (triangle, TetGen, NetGen ...) in order to obtain triangulation
- Performe finite volume or finite element discretization of the problem.

Alternative way:

- Construction "by hand" on regular structures

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Partial Differential Equations

DIfferential operators

- Bounded domain $\Omega \subset \mathbb{R}^d$, with piecewise smooth boundary
- Scalar function $u : \Omega \to \mathbb{R}$
- Vector function $\mathbf{v} : \Omega \to \mathbb{R}^d$
- Write $\partial_i u = \frac{\partial u}{x_i}$
- For a multindex $\alpha = (\alpha_1 \dots \alpha_d)$, write $|\alpha| = \alpha_1 + \dots + \alpha_d$ and define $\partial^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdot \dots \cdot \partial x_d^{\alpha_d}}$
- Gradient $\mathrm{grad} = \nabla$: $u \mapsto \nabla u = \begin{pmatrix} \partial_1 u \\ \vdots \\ \partial_d u \end{pmatrix}$
- Divergence $\mathrm{div} = \nabla \cdot$: $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \mapsto \nabla \cdot \mathbf{v} = \partial_1 v_1 + \dots + \partial_d v_d$
- Laplace operator $\Delta = \mathrm{div} \cdot \mathrm{grad} = \nabla \cdot \nabla$: $u \mapsto \Delta u = \partial_{11} u + \dots + \partial_{dd} u$

Lebesgue integral, $L^1(\Omega)$ I

- Let Ω have a boundary which can be represented by continuous, piecewiese smooth functions in local coordinate systems, without cusps and other+ degeneracies (more precisely: Lipschitz domain).
  - Polygonal domains are Lipschitz.
- Let $C_c(\Omega)$ be the set of continuous functions $f : \Omega \to \mathbb{R}$ with compact support.
- For these functions, the Riemann integral $\int_\Omega f(x)dx$ is well defined, and $||f|| := \int_\Omega |f(x)|dx$ provides a norm, and induces a metric
- A Cauchy sequence is a sequence f_n of functions where the norm of the difference between two elements can be made arbitrarily small by increasing the element numbers:

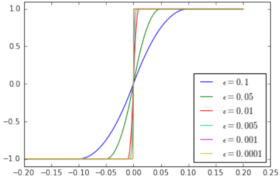
$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} : \forall m, n > n, ||f_n - f_m|| < \varepsilon$$

- All convergent sequences of functions are Cauchy sequences
- A metric space is *complete* if all Cauchy sequences of its element have a limit within this space

Lebesgue integral, $L^1(\Omega)$ II

- Let $L^1(\Omega)$ be the completion of $C_c(\Omega)$ with respect to the metric defined by the integral norm, i.e. "include" all limites of Cauchy sequences
- Defined via sequences, $\int_\Omega |f(x)| dx$ is defined for all functions in $L^1(\Omega)$.
- Equality of L^1 functions is elusive as they are not necessarily continuous: best what we can say is that they are equal "almost everywhere".
- Examples for Lebesgue integrable (measurable) functions:
  - Bounded functions continuous except in a finite number of points
  - Step functions

$$f_\epsilon(x) = \begin{cases} 1, & x \geq \epsilon \\ -(\frac{x-\epsilon}{\epsilon})^2 + 1, & 0 \leq x < \epsilon \\ (\frac{x+\epsilon}{\epsilon})^2 - 1, & -\epsilon \leq x < 0 \\ -1, & x < -\epsilon \end{cases} \xrightarrow{\epsilon \to 0} f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & \text{else} \end{cases}$$



Spaces of integrable functions

- For $1 \leq p \leq \infty$, let $L^p(\Omega)$ be the space of measureable functions such that

$$\int_\Omega |f(x)|^p dx < \infty$$

equipped with the norm

$$||f||_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

- These spaces are *Banach spaces*, i.e. complete, normed vector spaces.
- The space $L^2(\Omega)$ is a *Hilbert space*, i.e. a Banach space equipped with a scalar product $(\cdot, \cdot)$ whose norm is induced by that scalar product, i.e. $||u|| = \sqrt{(u, u)}$. The scalar product in L^2 is

$$(f, g) = \int_\Omega f(x)g(x)dx.$$

Green's theorem

- Green's theorem for *smooth* functions: Let $u, v \in C^1(\overline{\Omega})$ (continuously differentiable). Then for $\mathbf{n} = (n_1 \dots n_d)$ being the outward normal to Ω,

$$\int_\Omega u \partial_i v dx = \int_{\partial\Omega} u v n_i ds - \int_\Omega v \partial_i u dx$$

In particular, if $v = 0$ on $\partial\Omega$ one has

$$\int_\Omega u \partial_i v dx = - \int_\Omega v \partial_i u dx$$

Weak derivative

- Let $L^1_{loc}(\Omega)$ the set of functions which are Lebesgue integrable on every compact subset $K \subset \Omega$. Let $C_0^\infty(\Omega)$ be the set of functions infinitely differentiable with zero values on the boundary.

For $u \in L^1_{loc}(\Omega)$ we *define* $\partial_i u$ by

$$\int_\Omega v \partial_i u dx = -\int_\Omega u \partial_i v dx \quad \forall v \in C_0^\infty(\Omega)$$

and $\partial^\alpha u$ by

$$\int_\Omega v \partial^\alpha u dx = (-1)^{|\alpha|} \int_\Omega u \partial_i v dx \quad \forall v \in C_0^\infty(\Omega)$$

if these integrals exist.

Sobolev spaces

- For $k \geq 0$ and $1 \leq p < \infty$, the *Sobolev space* $W^{k,p}(\Omega)$ is the space functions where all up to the k-th derivatives are in L^p:

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \ \forall |\alpha| \leq k\}$$

with then norm

$$||u||_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} ||\partial^\alpha u||^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

- Alternatively, they can be defined as the completion of C^∞ in the norm $||u||_{W^{k,p}(\Omega)}$
- $W_0^{k,p}(\Omega)$ is the completion of C_0^∞ in the norm $||u||_{W^{k,p}(\Omega)}$
- The Sobolev spaces are Banach spaces.

Fractional Sobolev spaces and traces

- For $0 < s < 1$ define the *fractional Sobolev space*

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{u(x) - u(y)}{||x - y||^{s + \frac{d}{p}}} \in L^p(\Omega \times \Omega) \right\}$$

- Let $H^{\frac{1}{2}}(\Omega) = W^{\frac{1}{2},2}(\Omega)$
- A priori it is hard to say what the value of a function from L^p on the boundary is like.
- For Lipschitz domains there exists unique continuous *trace mapping* $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ such that
  - $\mathrm{Im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$
  - $\mathrm{Ker}\gamma_0 = W_0^{1,p}(\Omega)$

Sobolev spaces of square integrable functions

- $H^k(\Omega) = W^{k,2}(\Omega)$ with the scalar product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega \partial^\alpha u \partial^\alpha v \, dx$$

  is a Hilbert space.
- $H^k(\Omega)_0 = W_0^{k,2}(\Omega)$ with the scalar product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega \partial^\alpha u \partial^\alpha v \, dx$$

  is a Hilbert space as well.
- The initally most important:
  - $L^2(\Omega)$ with the scalar product $(u, v)_{L^2(\Omega)} = \int_\Omega uv \, dx$
  - $H^1(\Omega)$ with the scalar product $(u, v)_{H^1(\Omega)} = \int_\Omega (uv + \nabla u \cdot \nabla v) \, dx$
  - $H_0^1(\Omega)$ with the scalar product $(u, v)_{H_0^1(\Omega)} = \int_\Omega (\nabla u \cdot \nabla v) \, dx$
- Inequalities:

$$\begin{aligned} |(u, v)|^2 &\leq (u, u)(v, v) && \text{Cauchy-Schwarz} \\ ||u + v|| &\leq ||u|| + ||v|| && \text{Triangle inequality} \end{aligned}$$

Heat conduction revisited: Derivation of weak formulation

- Sobolev space theory provides the necessary framework to formulate existence and uniqueness of solutions of PDEs.
- Heat conduction equation with homogeneous Dirichlet boundary conditions:

$$
\begin{aligned}
-\nabla \cdot \lambda \nabla u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

Multiply and integrate with an arbitrary *test function* from $C_0^\infty(\Omega)$:

$$
\begin{aligned}
-\int_\Omega \nabla \cdot \lambda \nabla u v \, dx &= \int_\Omega f v \, dx \\
\int_\Omega \lambda \nabla u \nabla v \, dx &= \int_\Omega f v \, dx
\end{aligned}
$$

Weak formulation of homogeneous Dirichlet problem

- Search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, dx = \int_\Omega fv \, dx \;\forall v \in H_0^1(\Omega)$$

- Then,

$$a(u,v) := \int_\Omega \lambda \nabla u \nabla v \, dx$$

is a self-adjoint bilinear form defined on the Hilbert space $H_0^1(\Omega)$.

It is bounded due to Cauchy-Schwarz:

$$|a(u,v)| = |\lambda| |\int_\Omega \nabla u \nabla v \, dx| \leq ||u||_{H_0^1(\Omega)} \cdot ||v||_{H_0^1(\Omega)}$$

- $f(v) = \int_\Omega fv \, dx$ is a linear functional on $H_0^1(\Omega)$. For Hilbert spaces V the dual space V' (the space of linear functionals) can be identified with the space itself.

The Lax-Milgram lemma

Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive, i.e.

$$\exists \alpha > 0 : \forall u \in V, a(u,u) \geq \alpha ||u||_V^2.$$

Then the problem: find $u \in V$ such that

$$a(u,v) = f(v) \; \forall v \in V$$

admits one and only one solution with an a priori estimate

$$||u||_V \leq \frac{1}{\alpha} ||f||_{V'}$$

Heat conduction revisited

Let $\lambda > 0$. Then the weak formulation of the heat conduction problem: search $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \nabla v \, dx = \int_\Omega fv \, dx \; \forall v \in H_0^1(\Omega)$$

has an unique solution.

Proof: $a(u, v)$ is cocercive:

$$a(u, v) = \int_\Omega \lambda \nabla u \nabla u \, dx = \lambda ||u||^2_{H_0^1(\Omega)}$$

□

Weak formulation of inhomogeneous Dirichlet problem

$$-\nabla \cdot \lambda \nabla u = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

If g is smooth enough, there exists a *lifting* $u_g \in H^1(\Omega)$ such that $u_g|_{\partial\Omega} = g$. Then, we can re-formulate:

$$-\nabla \cdot \lambda \nabla (u - u_g) = f + \nabla \cdot \lambda \nabla u_g \text{ in } \Omega$$
$$u - u_g = 0 \text{ on } \partial\Omega$$

- Search $u \in H^1(\Omega)$ such that

$$u = u_g + \phi$$
$$\int_\Omega \lambda \nabla \phi \nabla v \, dx = \int_\Omega f v \, dx + \int_\Omega \lambda \nabla u_g \nabla v \quad \forall v \in H_0^1(\Omega)$$

Here, necessarily, $\phi \in H_0^1(\Omega)$ and we can apply the theory for the homogeneous Dirichlet problem.

Weak formulation of Robin problem

$$-\nabla \cdot \lambda \nabla u = f \text{ in } \Omega$$
$$\lambda \nabla u \cdot \mathbf{n} + \alpha(u - g) = 0 \text{ on } \partial\Omega$$

Multiply and integrate with an arbitrary *test function* from $C_c^\infty(\Omega)$:

$$-\int_\Omega (\nabla \cdot \lambda \nabla u) v \, dx = \int_\Omega fv \, dx$$
$$\int_\Omega \lambda \nabla u \nabla v \, dx + \int_{\partial\Omega} (\lambda \nabla u \cdot \mathbf{n}) v ds = \int_\Omega fv \, dx$$
$$\int_\Omega \lambda \nabla u \nabla v \, dx + \int_{\partial\Omega} \alpha uv \, ds = \int_\Omega fv \, dx + \int_{\partial\Omega} \alpha gv \, ds$$

Weak formulation of Robin problem II

- Let

$$a^R(u,v) := \int_\Omega \lambda \nabla u \nabla v \, dx + \int_{\partial\Omega} \alpha uv \, ds$$
$$f^R(v) := \int_\Omega fv \, dx + \int_{\partial\Omega} \alpha gv \, ds$$

  The integrals over $\partial\Omega$ must be understood in the sense of the trace space $H^{\frac{1}{2}}(\partial\Omega)$.

- Search $u \in H^1(\Omega)$ such that

$$a^R(u,v) = f^R(v) \; \forall v \in H^1(\Omega)$$

- If $\lambda > 0$ and $\alpha > 0$ then $a^R(u,v)$ is cocercive.

Neumann boundary conditions

Homogeneous Neumann:

$$\lambda \nabla u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega$$

Inhomogeneous Neumann:

$$\lambda \nabla u \cdot \mathbf{n} = g \text{ on } \partial\Omega$$

Weak formulation:

- Search $u \in H^1(\Omega)$ such that

$$\int_\omega \nabla u \nabla v dx = \int_{\partial\Omega} gvds \;\forall v \in H^1(\Omega)$$

Not coercive due to the fact that we can add an arbitrary constant to u and $a(u, u)$ stays the same!

Further discussion on boundary conditions

- Mixed boundary conditions:
  One can have differerent boundary conditions on different parts of the boundary. In particular, if Dirichlet or Robin boundary conditions are applied on at least a part of the boundary of measure larger than zero, the binlinear form becomes coercive.
- Natural boundary conditions: Robin, Neumann
  These are imposed in a "natural" way in the weak formulation
- Essential boundary conditions: Dirichlet
  Explicitely imposed on the function space
- Coefficients $\lambda, \alpha \ldots$ can be functions.

The Dirichlet penalty method

- Robin problem: search $u_\alpha \in H^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u_\alpha \nabla v \, dx + \int_{\partial\Omega} \alpha u_\alpha v \; ds = \int_\Omega fv \, dx + \int_{\partial\Omega} \alpha g v \, ds \forall v \in H^1(\Omega)$$

- Dirichlet problem: search $u \in H^1(\Omega)$ such that

$$u = u_g + \phi \quad \text{where } u_g|_{\partial\Omega} = g$$

$$\int_\Omega \lambda \nabla \phi \nabla v \, dx = \int_\Omega fv \, dx + \int_\Omega \lambda \nabla u_g \nabla v \;\; \forall v \in H_0^1(\Omega)$$

- Penalty limit:

$$\lim_{\alpha \to \infty} u_\alpha = u$$

- Formally, the convergence rate is quite low
- Implementing Dirichlet boundary conditions directly leads to a number of technical problems
- Implementing the penalty method is technically much simpler
- Proper way of handling the parameter leads to exact fulfillment of Dirichlet boundary condition in the floating point precision

The Galerkin method I

- Weak formulations “live” in Hilbert spaces which essentially are infinite dimensional
- For computer representations we need finite dimensional approximations
- The finite volume method provides one possible framework which in many cases is close to physical intuition. However, its error analysis is hard.
- The Galerkin method and its modifications provide a general scheme for the derivation of finite dimensional appoximations

The Galerkin method II

- Let V be a Hilbert space. Let $a : V \times V \to \mathbb{R}$ be a self-adjoint bilinear form, and f a linear functional on V. Assume a is coercive with coercivity constant α, and continuity constant γ.
- Continuous problem: search $u \in V$ such that

$$a(u, v) = f(v) \ \forall v \in V$$

- Let $V_h \subset V$ be a finite dimensional subspace of V
- "Discrete" problem $\equiv$ Galerkin approximation:
  Search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \ \forall v_h \in V_h$$

By Lax-Milgram, this problem has a unique solution as well.

Céa's lemma

- What is the connection between u and u_h ?
- Let $v_h \in V_h$ be arbitrary. Then

$$\begin{aligned}
\alpha||u - u_h||^2 &\leq a(u - u_h, u - u_h) \quad \text{(Coercivity)} \\
&= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
&= a(u - u_h, u - v_h) \quad \text{(Galerkin Orthogonality)} \\
&\leq \gamma||u - u_h|| \cdot ||u - v_h|| \quad \text{(Boundedness)}
\end{aligned}$$

- As a result

$$||u - u_h|| \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u - v_h||$$

- Up to a constant, the error of the Galerkin approximation is the error of the best approximation of the solution in the subspace V_h.

From the Galerkin method to the matrix equation

- Let $\phi_1 \dots \phi_n$ be a set of basis functions of V_h.
- Then, we have the representation $u_h = \sum_{j=1}^{n} u_j \phi_j$
- In order to search $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \ \forall v_h \in V_h$$

  it is actually sufficient to require

$$\begin{aligned} a(u_h, \phi_i) &= f(\phi_i) \ (i = 1 \dots n) \\ a\left(\sum_{j=1}^{n} u_j \phi_j, \phi_i\right) &= f(\phi_i) \ (i = 1 \dots n) \\ \sum_{j=1}^{n} a(\phi_j, \phi_i) u_j &= f(\phi_i) \ (i = 1 \dots n) \\ AU &= F \end{aligned}$$

  with $A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j)$, $F = (f_i), f_i = F(\phi_i)$, $U = (u_i)$.
- Matrix dimension is $n \times n$. Matrix sparsity ?

Obtaining a finite dimensional subspace

- Let $\Omega = (a, b) \subset \mathbb{R}^1$
- Let $a(u, v) = \int_\Omega \lambda(x) \nabla u \nabla v dx$.
- Analysis I provides a finite dimensional subspace: the space of sin/cos functions up to a certain frequency $\Rightarrow$ *spectral method*
- Ansatz functions have global support $\Rightarrow$ full $n \times n$ matrix
- OTOH: rather fast convergence for smooth data
- Generalization to higher dimensions possible
- Big problem in irregular domains: we need the eigenfunction basis of some operator. . .
- Spectral methods are successful in cases where one has regular geometry structures and smooth/constant coefficients – e.g. "Spectral Einstein Code"

The finite element idea

- Choose basis functions with local support. In this case, the matrix becomes sparse, as only integrals of basis function pairs with overlapping support contribute to the matrix.
- Linear finite elements in $\Omega = (a, b) \subset \mathbb{R}^1$:
- Partition $a = x_1 \leq x_2 \leq \cdots \leq x_n = b$
- Basis functions (for $i = 1 \dots n$)

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & i > 1, x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & i < n, x \in (x_i, x_{i+1}) \\ 0, & \text{else} \end{cases}$$

- Any function $u_h \in V_h = \mathrm{span}\{\phi_1 \dots \phi_n\}$ is piecewise linear, and the coefficients in the representation $u_h = \sum_{i=1}^n u_i \phi_i$ are the values $u_h(x_i)$.
- Fortunately, we are working with a weak formulation, and weak derivatives are well defined !

1D matrix elements

($\lambda = 1$, $x_{i+1} - x_i = h$) - The integrals are nonzero for $i = j, i+1 = j, i-1 = j$
Let $j = i + 1$

$$
\begin{aligned}
a_{ij} = a(\phi_i, \phi_{i+1}) &= \int_\Omega \nabla\phi_i \nabla\phi_j dx = \int_{x_i}^{x_{i+1}} \nabla\phi_i \nabla\phi_j dx = -\int_{x_i}^{x_{i+1}} \frac{1}{h^2} dx \\
&= \frac{1}{h} dx
\end{aligned}
$$

Similarly, $a(\phi_i, \phi_{i-1}) = -\frac{1}{h}$

For $1 < i < N$:

$$
\begin{aligned}
a_{ii} = a(\phi_i, \phi_i) &= \int_\Omega \nabla\phi_i \nabla\phi_i dx = \int_{x_{i-1}}^{x_{i+1}} \nabla\phi_i \nabla\phi_i dx = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} dx \\
&= \frac{2}{h} dx
\end{aligned}
$$

For $i = 1$ or $i = N$, $a(\phi_i, \phi_i) = \frac{1}{h}$

1D matrix elements II

Adding the boundary integrals yields

$$A = \begin{pmatrix} \alpha + \frac{1}{h} & -\frac{1}{h} & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{1}{h} + \alpha \end{pmatrix}$$

... the same matrix as for the finite volume method...

Where to go from here

- Higher space dimensions
- Piecewise polynomials instead of piecewise linear functions