Some applications of cubature on Wiener space

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February 8, 2011
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Outline

Weak approximation of solutions of SDEs

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i =: \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i, \qquad (1)$$

- $V_0, \ldots, V_d : \mathbb{R}^N \to \mathbb{R}^N$ vector fields,
- B_t a d-dimensional Brownian motion, $B_t^0 := t$,
- $X_0 = x \in \mathbb{R}^N$.

Problem

For $f : \mathbb{R}^N \to \mathbb{R}$ sufficiently regular, compute $u(0, x) := E\left[f(X_T)\right]$.

Example

- Option pricing
- Numerical solution of parabolic PDEs: $\partial_t u + Lu = 0$

Stochastic Taylor expansion

Ito formula for Stratonovich calculus

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where $V_i f(x) := V_i(x) \cdot \nabla f(x)$.

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Stochastic Taylor expansion

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$$f(X_t) = f(x) + \sum_{i_1=0}^{d} V_{i_1} f(x) B_t^{i_1} + \sum_{i_1,i_2=0}^{d} V_{i_1} V_{i_2} f(x) \int_0^t B_{t_2}^{i_1} \circ dB_{t_2}^{i_2}$$
$$+ \sum_{i_1,i_2,i_3=0}^{d} \int_{0\le t_1 \le t_2 \le t_3 \le t} V_{i_1} V_{i_2} V_{i_3} f(X_{t_1}) \circ dB_{t_1}^{i_1} \circ dB_{t_2}^{i_2} \circ dB_{t_3}^{i_3},$$

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Stochastic Taylor expansion

$$f(X_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1} \cdots V_{i_k} f(x) B_t^{(i_1,\dots,i_k)} + R_m(t,x,f),$$

$\sup_x \sqrt{E\left[R_m^2\right]} = \mathcal{O}(t^{\frac{m+1}{2}})$, $B_t^{(i_1,\dots,i_k)} := \int_{0\le t_1\le\cdots\le t_k\le t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k}$.

Random ODEs

- Let W be a $(d+1)$-dimensional process with paths of bounded variation, define $\widetilde{X}_t = X(W)_t$ by the random ODE

$$\frac{d}{dt}\widetilde{X}_t = \sum_{i=0}^{d} V_i(\widetilde{X}_t)\dot{W}_t^i, \quad \widetilde{X}_0 = x. \tag{2}$$

- Ordinary Taylor expansion:

$$f(\widetilde{X}_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1}\cdots V_{i_k} f(x) W_t^{(i_1,\dots,i_k)} + \widetilde{R}_m(t,x,f)$$

- Remember: Stochastic Taylor expansion

$$f(X_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1}\cdots V_{i_k} f(x) B_t^{(i_1,\dots,i_k)} + R_m(t,x,f)$$

Cubature on Wiener space

Definition

W is a cubature formula on Wiener space of degree m iff $E\left[W_t^{(i_1,\ldots,i_k)}\right] = E\left[B_t^{(i_1,\ldots,i_k)}\right]$ for $k \leq m$.

Remark

In fact, only need multi-indices $(i_1, \ldots, i_k)$ such that $k + \#\{\ell : i_\ell = 0\} \leq m$: B_t^0 counts twice due to scaling of Brownian motion. This property is ignored for ease of presentation!

- Cubature formulas with finite support exist (Lyons and Victoir)
- Construction of cubature formulas for $m > 5$ interesting open problem

Weak approximation

- Local error: $E\left[f(X_t)\right] - E\left[f(\widetilde{X}_t^{(m)})\right] = \mathcal{O}(t^{(m+1)/2})$.
- Fix a grid $0 = t_0 < t_1 < \cdots < t_n = T$, define W by concatenation of independent cubature formulas (of degree m) on the subintervals $[t_i, t_{i+1}]$.
- Global error: $E\left[f(X_T)\right] - E\left[f(\widetilde{X}_T^{(m)})\right] = \mathcal{O}((\sup \Delta t)^{(m-1)/2})$
- Used very stringent regularity conditions!
- Weaker assumption: non-uniform grid + Hörmander condition, allow f to be uniformly Lipschitz only (Kusuoka)
- Support of W grows exponentially in n, but n usually small (otherwise: recombination techniques or (quasi) Monte Carlo)

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Extensions

- Jump diffusions (B. and Teichmann): add jump times to grid, but at most $m/2$ for each initial interval
- May reduce order of cubature method by two for each jump
- Backward SDEs (Crisan and Manolarakis): allows to solve semilinear parabolic PDEs with cubature methods
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Example: SPDEs

$$dX_t = (AX_t + \alpha(X_t))dt + \sum_{i=1}^{d} \sigma_i(X_t)dB_t^i$$

- Cubature method: solution $\widetilde{X}$ of random PDE

$$\dot{\widetilde{X}}_t = (A\widetilde{X}_t + \alpha_0(\widetilde{X}_t)) + \sum_{i=1}^{d} \sigma_i(\widetilde{X}_t)\dot{W}_t^i$$

- Use existing deterministic PDE solvers
- Euler method: iteration

$$\overline{X}_{n+1} = (A\overline{X}_n + \alpha(\overline{X}_n))\Delta t_n + \sum_{i=1}^{d} \sigma_i(\overline{X}_n)\Delta B_n^i,$$

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Signature

Definition

The collection of random variables

$$S^m_{0,t} := \left(B_t^{(i_1,\dots,i_k)} \right)_{k \le m}$$

is called truncated signature of the Brownian motion. (Analogous definition for other processes/paths.)

- Values in a certain step m nilpotent Lie group ($m = 2$: Heisenberg group).
- Algebra of paths corresponds nicely to group structure:
  - concatenation of paths ≡ multiplication of signatures
  - scaling of paths ≡ dilatation on the group
  - metric on group obtained via geodesic paths
- Rough path: $1/p$-Hölder continuous path in the Lie group. (For Brownian motion: $2 < p < 3$.)

Universal limit theorem

- Rough DE: for a *d*-dimensional (smooth) path *w* consider

$$dX(w)_t = \sum_{i=0}^{d} V_i(X(w)_t)dw_t^i$$

- Define the Ito map $I(S_{0,\cdot}^m(w)) := X(w)$.

Theorem (Lyons)

I is continuous in $1/p$-Hölder topology, uniformly on bounded sets, provided that $m \geq \lfloor p \rfloor$. Thus, the solution to the rough equation can be extended to a wide family of non-smooth paths (+ signatures).

- Solution of SDE is continuous map of Brownian motion and Lévy area.
- Given processes W^n s.t. $S(W^n) \to S(B)$, we have strong convergence of the solutions of SDEs. E.g., Wong-Zakai theorem.

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Approximation

- For Continuity, only $m = 2$ needed, but higher order signature gives better approximation
- Assume a grid $0 = t_0 < t_1 < \cdots < t_n = T$ and a piecewise smooth process W on the grid such that $S^m(W)_{t_i,t_{i+1}} = S^m(B)_{t_i,t_{i+1}}$.
- Approximation: $\left|I(W) - I(B)\right| \leq (\sup_i \Delta t_i)^{(m+1-p)/p}$, $2 < p < 3$.
- Requires sampling of Lévy area.
- Cubature on Wiener space: Weak version of this result.

Path-dependent functionals (B. and Friz)

Let f be a continuous functional on paths. Goal: Compute $E[f(X_\cdot)]$ using cubature on Wiener space.

- Consider a grid $0 = t_0 < \cdots < t_n = T$ and a cubature formula W on the grid.
- By a Donsker theorem for processes with Hölder paths in the Heisenberg group, $S^2(W)_{0,\cdot}$ converges weakly to $S^2(B)_{0,\cdot}$.
- By the universal limit theorem, this implies convergence

$$E\left[f(\widetilde{X}_\cdot)\right] \to E\left[f(X_\cdot)\right]$$

when $\sup \Delta t_i \to 0$.

The Ninomiya-Victoir method

- On a (uniform) grid $0 = t_0 < \cdots < t_n = T$ set $\Delta t_i := t_{i+1} - t_i$, $\Delta B^j_i := B^j_{t_{i+1}} - B^j_{t_i}$, Λ_i Bernoulli-distributed
- Set $\overline{X}_0 = x$ and iteratively

$$\overline{X}_{i+1} := \begin{cases} e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B^d_i V_d} \cdots e^{\Delta B^1_i V_1} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = 1, \\ e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B^1_i V_1} \cdots e^{\Delta B^d_i V_d} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = -1. \end{cases} \quad (3)$$

- $e^{sV_i} x := z(1)$, where $\dot{z}(t) = sV_i(z(t))$, $z(0) = x$
- Global error: $E\left[f(X_T)\right] - E\left[f\left(\overline{X}_n\right)\right] = O((\sup \Delta t_i)^2)$
- Interpretation as cubature method and splitting method

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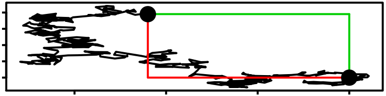

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$$Q_{\Delta t}^{NV} = \frac{1}{2} e^{\frac{\Delta t}{2} L_0} e^{\Delta t L_1} \cdots e^{\Delta t L_d} e^{\frac{\Delta t}{2} L_0} + \frac{1}{2} e^{\frac{\Delta t}{2} L_0} e^{\Delta t L_d} \cdots e^{\Delta t L_1} e^{\frac{\Delta t}{2} L_0},$$

where $L_0 f(x) = V_0 f(x)$, $L_i f(x) = \frac{1}{2} V_i^2 f(x)$,
$Q_{\Delta t}^{NV} \approx P_{\Delta t} := e^{\Delta t L_0 + \Delta t \sum_{i=1}^d L_i}$

Applicability in finance (B., Friz and Loeffen)

- Very advantageous when all ODEs can be solved explicitly (otherwise: can use high order Runge-Kutta schemes).

Example (Generalized SABR model)

$$dX_t^1 = a\left(X_t^2\right)^{\alpha}\left(X_t^1\right)^{\beta} dB_t^1,$$

$$dX_t^2 = \kappa(\theta - X_t^2)dt + bX_t^2(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2),$$

where $1/2 \leq \alpha, \beta \leq 1$. (SABR: $\alpha = 1, \kappa = 0$.)

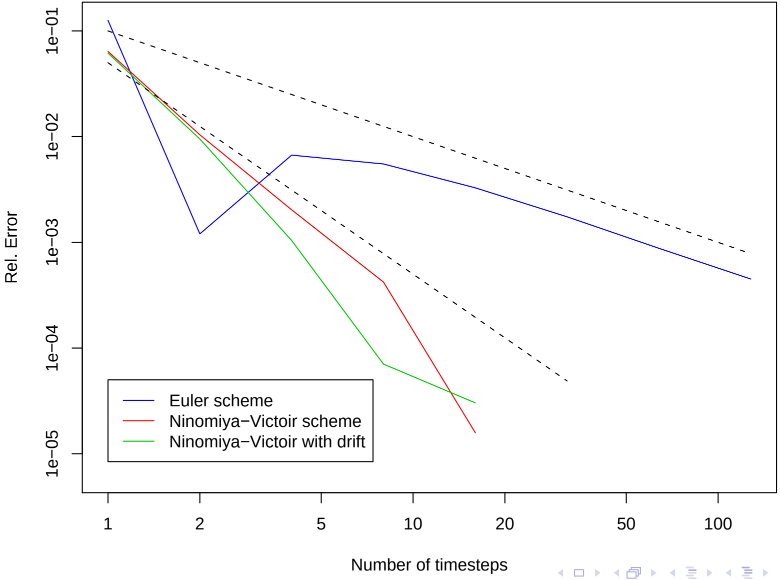
Example: Generalized SABR model – continued

- *Drift trick:* choose $\gamma \in \mathbb{R}^d$, set $V_0^{(\gamma)}(x) := V_0(x) - \sum_{i=1}^d \gamma^i V_i(x)$ and consider

$$dX_t = V_0^{(\gamma)}(X_t)dt + \sum_{i=1}^d V_i(X_t) \circ d\left(B_t^i + \gamma^i t\right)$$

- Apply N-V-scheme for vector fields $V_0^{(\gamma)}, V_1, \dots, V_d$ with ΔB_j^i replaced by $\Delta B_j^i + \gamma^i \Delta t$

Generalized SABR – Numerical experiment



Generalized SABR – Computational time

Method	K	M	Rel. Error	Time
Euler	32	8192000	0.00174	91.94 sec
Ninomiya-Victoir	4	2048000	0.00204	13.93 sec
NV with drift	4	1024000	0.00104	2.88 sec

Multi-Level Monte Carlo (Giles)

- Systematic variance reduction technique
- $\overline{X}_T^{(n)} \approx X_T$ based on a uniform grid with size n.
- Idea:
  1. Use $\overline{X}_T^{(n/2)}$ as control variate for $\overline{X}_T^{(n)}$; requires computation of $E\left[\overline{X}_T^{(n/2)}\right]$ with high accuracy.
  2. Use $\overline{X}_T^{(n/4)}$ as control variate for $\overline{X}_T^{(n/2)}$;
  3. ...
- Optimal: Work at each level is equal, i.e., the finer the grid, the fewer samples need to be simulated.
- Time discretization error depends on finest grid, (Monte Carlo) integration error on coarsest grid (with most samples).

Example

Euler method: complexity reduced from $O(\epsilon^{-3})$ to $O(\epsilon^{-2}(\log \epsilon)^2)$.

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Error control

Idea

Want to use a fine grid only when/where quantity of interest is sensitive.

- Need some computable error control
- A priori error estimates: require no/little additional computations, but are very crude.
- A posteriori estimates: possibly substantial additional work, but accurate error control.
- Computable a posteriori estimates available following Talay and Tubaro

Adaptivity following Szepessy et al

(Stochastic) Control problem

Minimize (expected) work subject to the error (estimate) being smaller than TOL.
Control variable: grid

- Deterministic control problem: leads to non-uniform, deterministic grid
- Stochastic control problem: leads to non-uniform, random grid

Stochastic Algorithm

- Start with coarse grid, compute error estimate
- Where necessary, refine grid, and iterate
- Refinement requires some bridging procedure

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