Existence, Uniqueness and Stability of Invariant Distributions in Continuous-Time Stochastic Models

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SAET 2012 Conference Brisbane
July 1, 2012

Outline

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Search and matching models

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- Extremely popular these days (and for many years now)
- Inspired by work of Diamond, Mortensen and Pissarides
- Apart from some recent examples (e.g. our companion paper and the references therein), these models do not include a saving mechanism

Our setup

- Pissarides textbook model (without Nash-bargaining) extended for a consumption-saving mechanism
- Process of matching and separation is augmented to allow for self-insurance of workers
- We describe distributional prediction for labour market status and wealth using Fokker-Planck equations in a companion paper

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The model

Matching on the labour market

- Transitions between states $z \in \{w, b\}$ with (state-dependent) matching rate μ and separation rate s
- Wage w and benefits b are exogenous (in this stability paper, not in companion paper)
- Representation for maximisation problem as a stochastic differential equation with two Poisson processes

$$dz(t) = \Delta \left[dq_\mu - dq_s \right], \quad \Delta \equiv w - b$$

- Corresponds to cont. time Markov chain

Budget constraint of an individual

$$da(t) = \{ra(t) + z(t) - c(t)\} dt$$

- Interest rate on wealth r, consumption $c(t)$

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Optimality

Utility functions

- Intertemporal

$$U(t) = E_t \int_t^{\infty} e^{-\rho[\tau - t]} u(c(\tau))\, d\tau$$

- CRRA instantaneous utility function

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

Optimality condition

- Generalized Keynes-Ramsey rule
- Represented for this paper by policy function $c(a(t), z(t))$

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Dynamics

System to be understood

- Frictional labour market equation

$$dz(t) = \Delta\left[dq_\mu - dq_s\right], \quad \Delta \equiv w - b$$

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Different regimes

- Low interest rate $r \leq \rho$: bounded state space $[-b/r, a_w^*]$ for wealth
- High interest rate $r \geq \rho + \mu$: a_t increasing to ∞
- Intermediate case: a_t increasing to ∞ when larger than a threshold value

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Setting

- State space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ locally compact separable metric space
- $(X_t)_{t\in[0,\infty[}$ right-continuous, time-homogeneous strong Markov process
- *Transition kernel* $P^t(x, A) := P(X_t \in A | X_0 = x)$
- *Semi-group* $P_t f(x) := E\left[f(X_t) | X_0 = x\right] = \int_{\mathbf{X}} f(y) P^t(x, dy)$.

Example

For the wealth-employment process (A_t, z_t) in the low-interest-regime, the state space is chosen to be $\mathbf{X} = [-b/r, a_w^*] \times \{w, b\}$, a compact, separable metric space.

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Methodology

There are (at least) two very different approaches:

Functional analysis: use the classical theory of strongly continuous semi-groups of linear operators on Banach spaces

Probability: analogy to discrete-time Markov chains, i.e., study the recurrence structure

- We are going to follow the probabilistic road, the semi-group $(P_t)_{t\in[0,\infty[}$ and its infinitesimal generator will *not* be used.
- Based on a long history of results, ultimate treatment by Meyn and Tweedie and their co-authors in 90's.

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Goal

- Stability is a rather vague concept.
- Here: ergodicity in the sense that for any initial state x, $P^t(x, \cdot) \xrightarrow{t\to\infty} \pi$ for some unique probability distribution π.
- No time-averaging necessary.

Definition

A measure μ on $\mathbf{X}$ is called invariant, iff

$$\forall A \in \mathcal{B}(\mathbf{X}),\ \forall t \geq 0 : P^t_\mu(A) := \int_{\mathbf{X}} P^t(x, A)\mu(dx) = \mu(A),$$

i.e., the process X_t with law$(X_0) = \mu$ is stationary.

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Outline of the proof of stability:

(1) Existence of an invariant distribution

(2) Uniqueness of invariant measures

(3) Convergence

Remark

(1) and (2) are very different in nature and rely on qualitatively different assumptions.

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Existence of an invariant probability measure

- Existence of an invariant probability distribution depends on a growth condition: no mass is allowed to escape to infinity.
- X_t is bounded in probability on average, iff $\forall x \in \mathbf{X},\ \epsilon > 0$ there is a compact set $C \subset \mathbf{X}$ s.t.

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t P^s(x, C)ds \geq 1 - \epsilon.$$

- Compactness of measures $\frac{1}{t} \int_0^t P^s(x, C)ds$
- X_t has the weak Feller property, iff for any bounded cont. $f : \mathbf{X} \to \mathbb{R}$ and $t > 0$, $x \mapsto \int_{\mathbf{X}} f(y)P^t(x, dy)$ is continuous.

Theorem (Beneš '68)

If the process X_t is bounded in probability on average and has the weak Feller property, then there is an invariant probability measure.

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[[SO EXISTENCE FOLLOWS FROM COMPACTNESS, AS THE ABOVE FAMILY OF MEASURES HAS SOME CONVERGENT SUBSEQUENCES. EACH LIMIT ALONG A CONVERGENT SUBSEQUENCE IS AN INVARIANT PROBABILITY MEASURE, BUT THERE CAN BE MORE THAN ONE.
CONCEPTUALLY, WHY DO WE EVEN NEED WEAK FELLER?
UNDERSTAND THE FOLLOWING PROOF!]]

Uniqueness of the invariant measure

- X_t is recurrent, iff there is a (non-trivial) σ-finite measure μ such that

$$A \in \mathcal{B}(\mathbf{X}),\ \mu(A) > 0 \Rightarrow \forall x \in \mathbf{X}:\ P\left(\tau_A < \infty | X_0 = x\right) = 1,$$

where $\tau_A := \inf\{t \geq 0 | X_t \in A\}$.

Theorem (Azéma, Duflo, Revuz '69)

If the process X_t is recurrent, then there is a unique σ-finite invariant measure (up to constant multiples).

Example

Let W_t be 1-dimensional Brownian motion. By recurrence, there is a unique invariant measure, whose density satisfies $\Delta f = 0$, implying that $f = 1$.

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[[UNDERSTAND WHY STRONG FELLER IMPLIES RECURRENCE/UNIQUENESS]]
[[UNDERSTAND WHY A-PERIODICITY NOT NECESSARY]]

Stability

- Stability for us means convergence $P^t(x,\cdot) \to \pi$ for any x in total variation, i.e.,

$$d_{TV}(P^t(x,\cdot),\pi) := \sup\left\{\left|P^t(x,A) - \pi(A)\right| \;\middle|\; A \in \mathcal{B}(\mathbf{X})\right\} \xrightarrow{t\to\infty} 0.$$

- Stability holds for a Harris recurrent Markov process X_t iff for some $\Delta > 0$, the skeleton chain $(X_{n\Delta})_{n\in\mathbb{N}}$ is irreducible.

Remark

Techniques based on Lyapunov functions even allow to specify the speed of convergence. But no general way to construct good Lyapunov functions.

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Existence and stability

- In the regimes of high and intermediate interest rates, wealth can converge to ∞.
- We concentrate on the low-interest-regime, where wealth is concentrated in a compact interval $[-b/r, a_w^*]$.
- By continuity of solutions of ODEs in the initial value, (a_t, z_t) is a continuous function of (a_0, z_0), implying the weak Feller property.
- Existence of invariant probability measures.
- If $z_0 = b$ or $z_0 = w$ and no jump, then $a_t \to -b/r$ or $a_t \to a_w^*$, respectively, implying the existence of an irreducible skeleton.
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Sufficient condition for recurrence

- Recurrence holds when transition kernel is smoothing.
- Diffusion case: recurrence follows under weak conditions
- Jump processes cannot smoothen as long as there is a positive probability of no jumps before t

Definition & theorem (Meyn and Tweedie '93)

X_t is called a T-process if there is a Markov kernel T and a prob. measure ν on $[0, \infty[$ s.t.

- $\forall A \in \mathcal{B}(\mathbf{X}) : \ x \mapsto T(x, A)$ is continuous
- $K_\nu(x, A) := \int_0^\infty P^t(x, A)\nu(dt) \geq T(x, A)$
- $\forall x : \ T(x, \mathbf{X}) > 0.$

Any irreducible T-process, which is bounded in probability on average, is recurrent.

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The wealth-employement process is a T-process

- Given (a_0, z_0), (a_t, z_t) is a deterministic function of the jump-times of z_t.
- Conditional on the number of jumps, the jump times have smooth densities.
- (a_t, z_t) is not smoothing, because no jump might occur.
- If at least one jump occurs, we have smoothing properties.
- Choose $\nu = \delta_\tau$ and

$$T((a_0, z_0), A) := P\Big((a_\tau, z_\tau) \in A, \text{ one jump in } [0, \tau] \,\Big|\, a_0, \; z_0\Big).$$

- Technical condition: $c = c(a, z)$ is C^1.
- Illustration of how T-property replaces strong Feller condition.

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- Individual maximization problem inspired by search and matching models
- Extended for consumption-saving problem
- Question: Is there a unique long-run distribution to which initial distributions converge?

Techniques

- Markov chain-style ergodicity analysis for general, continuous time Markov processes
- T-processes by Meyn and Tweedie allow to prove recurrence for a wide class of (degenerate) diffusion and jump models

Result

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