Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Smoothing the payoff for efficient computation of basket option prices

Christian Bayer

Markus Siebenmorgen, Raul Tempone

MCqMC 2018, Rennes

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 0 · www.wias-berlin.de

July 3, 2018

Consider a basket option on stocks (with $r = 0$, under Q)

$$S_T^i = S_0^i \exp\left(\sigma_i W_T^i - \frac{1}{2}\sigma_i^2 T\right), \quad i = 1, \dots, d, \quad T > 0,$$

i.e., we want to compute

$$C_{\mathcal{B}}(T, K) := E\left[\left(\sum_{i=1}^{d} c_i S_T^i - K\right)^+\right].$$

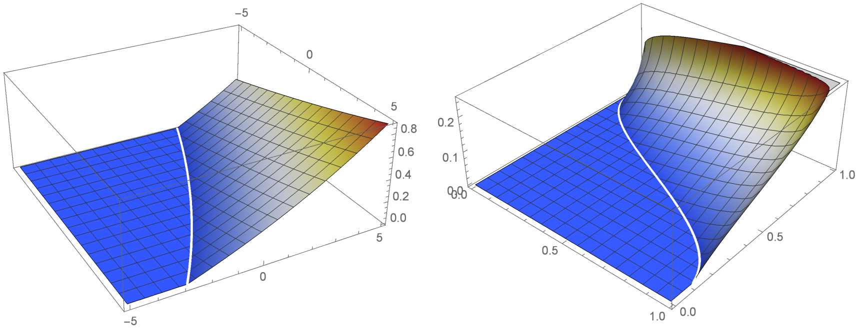
- $(W^1, \dots, W^d)$ is a correlated Brownian motion.
- $\sum_{i=1}^{d} c_i S_T^i$ is not lognormal.
- Solution methods:
  - Asymptotic formulae
  - Numerical integration

$$C_{\mathcal{B}} = E\left[\left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} Z_j\right) - K\right)^+\right]$$
$$= \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} z_j\right) - K\right)^+ \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|z|^2}{2}\right) dz$$
$$= \int_{[0,1]^d} \left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} \Phi^{-1}(u_j)\right) - K\right)^+ du$$

where $A A^\top = \Sigma$, $\Sigma_{ij} = \sigma_i \rho_{ij} \sigma_j T$.

- Cubature
  - Tensorized 1-dimensional quadrature
  - Sparse grid cubature
  - Multivariate cubature
- Quasi Monte Carlo
- Monte Carlo

Nonsmoothness of payoff



Plot of the payoff function ($d = 2$).

Left: $z \mapsto \left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} z_j\right) - K\right)^+$

Right: $u \mapsto \left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} \Phi^{-1}(u_j)\right) - K\right)^+$

Outline

Unbiased smoothing by conditional expectation

$$C_{\mathcal{B}} = E\left[\left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} Z_j\right) - K\right)^+\right]$$

- Suppose that $a_{i1} \equiv \alpha$

$$C_{\mathcal{B}} = E\left[\left(e^{\alpha Z_1} \sum_{i=1}^{d} w_i \exp\left(\sum_{j=2}^{d} a_{ij} Z_j\right) - K\right)^+\right]$$

- Conditioning on $Z_2, \ldots, Z_d$, we obtain

$$C_{\mathcal{B}} = E\left[C_{BS}\left(e^{\alpha^2/2} \sum_{i=1}^{d} w_i \exp\left(\sum_{j=2}^{d} a_{ij} Z_j\right), \alpha, K\right)\right],$$

$$C_{BS}(S_0, \sigma, K) := E\left[\left(S_0 e^{\sigma Z_1 - \sigma^2/2} - K\right)^+\right] = S_0 \Phi(d_1) - K\Phi(d_2),$$

where $d_{1/2} := \frac{1}{\sigma}\left(\log\left(\frac{S_0}{K}\right) \pm \frac{\sigma^2}{2}\right)$.

Note: C_{BS} is analytic in all its arguments provided $\sigma^2 > 0$.

Unbiased smoothing by conditional expectation

$$C_{\mathcal{B}} = E\left[\left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} Z_j\right) - K\right)^+\right]$$

- Suppose that $a_{i1} \equiv \alpha$

$$C_{\mathcal{B}} = E\left[\left(e^{\alpha Z_1} \sum_{i=1}^{d} w_i \exp\left(\sum_{j=2}^{d} a_{ij} Z_j\right) - K\right)^+\right]$$

- Conditioning on $Z_2, \ldots, Z_d$, we obtain

$$C_{\mathcal{B}} = E\left[C_{BS}\left(e^{\alpha^2/2} \sum_{i=1}^{d} w_i \exp\left(\sum_{j=2}^{d} a_{ij} Z_j\right), \alpha, K\right)\right],$$

$$C_{BS}(S_0, \sigma, K) := E\left[\left(S_0 e^{\sigma Z_1 - \sigma^2/2} - K\right)^+\right] = S_0 \Phi(d_1) - K\Phi(d_2),$$

where $d_{1/2} := \frac{1}{\sigma}\left(\log\left(\frac{S_0}{K}\right) \pm \frac{\sigma^2}{2}\right)$.

Note: C_{BS} is analytic in all its arguments provided $\sigma^2 > 0$.

Unbiased smoothing by conditional expectation

$$C_{\mathcal{B}} = E\left[\left(\sum_{i=1}^{d} w_i \exp\left(\sum_{j=1}^{d} a_{ij} Z_j\right) - K\right)^{+}\right]$$

- Suppose that $a_{i1} \equiv \alpha$

$$C_{\mathcal{B}} = E\left[\left(e^{\alpha Z_1} \sum_{i=1}^{d} w_i \exp\left(\sum_{j=2}^{d} a_{ij} Z_j\right) - K\right)^{+}\right]$$

- Conditioning on $Z_2, \ldots, Z_d$, we obtain

$$C_{\mathcal{B}} = E\left[C_{BS}\left(e^{\alpha^2/2} \sum_{i=1}^{d} w_i \exp\left(\sum_{j=2}^{d} a_{ij} Z_j\right), \alpha, K\right)\right],$$

$$C_{BS}(S_0, \sigma, K) \coloneqq E\left[\left(S_0 e^{\sigma Z_1 - \sigma^2/2} - K\right)^{+}\right] = S_0 \Phi(d_1) - K\Phi(d_2),$$

where $d_{1/2} \coloneqq \frac{1}{\sigma}\left(\log\left(\frac{S_0}{K}\right) \pm \frac{\sigma^2}{2}\right)$.

Note: C_{BS} is analytic in all its arguments provided $\sigma^2 > 0$.

Lemma

Let $\Sigma \in \mathbb{R}^{d \times d}$ symmetric, positive definite, $v \in \mathbb{R}^d$. Then there is a diagonal matrix $D = \operatorname{diag}\left(\lambda_1^2, \dots, \lambda_d^2\right)$ and an invertible matrix V with

- $\Sigma = VDV^\top$,
- $V_{i1} = v_i$, $i = 1, \dots, d$.

Proof.

1. Denote $w := \Sigma^{-1} v$. The matrix
$$\widetilde{\Sigma} := \Sigma - \frac{v \cdot v^\top}{v^\top \cdot w}$$
is symmetric, positive semidefinite with rank $d - 1$.
2. Denote $\lambda_i^2 > 0$ and $v_i \in \mathbb{R}^d$, $i = 2, \dots, d$, the positive eigenvalues of $\widetilde{\Sigma}$ and the corresponding eigenvectors and construct $V := (v, v_2, \dots, v_d)$, $\lambda_1^2 := (v^\top \cdot w)^{-1}$.

□

Theorem

Let $w_i := c_i S_0^i e^{-\sigma_i^2 T}$, $\Sigma_{ij} := \sigma_i \sigma_j \rho_{ij} T$, $\Sigma = VDV^\top$ *the factorization from the lemma with* $v = (1, \ldots, 1)^\top$, $Z \sim \mathcal{N}(0, I_{d-1})$, $\sqrt{D} := \mathrm{diag}(\lambda_2, \ldots, \lambda_d)$. *Then*

$$C_{\mathcal{B}} = E\left[C_{BS}\left(h(\sqrt{D}Z)e^{\lambda_1^2/2}, K, \lambda_1\right)\right],$$

$$h(y_2, \ldots, y_d) := \sum_{i=1}^{d} w_i \exp\left(\sum_{j=2}^{d} V_{ij} y_j\right).$$

- Explicit formula available as long as $v \in \{0, 1\}^d \setminus \{0\}$.
- Mollified payoff available in closed form and no bias introduced.
- Leads to reduced variance.
- Compare with domain transformation approaches (e.g., Achtsis, Cools, Nuyens '13)

Outline

$$\int_{\mathbb{R}^d} f(x)\varphi_d(x)dx = ?, \quad \varphi_d \dots \text{ standard normal density}$$

- Q_j, $j \in \mathbb{N}$, sequence of 1-dim. (Gaussian) quadrature formulas:

$$Q_j(g) = \sum_{\ell=1}^{N_j} v_\ell^{(j)} g(x_\ell^{(j)}), \quad \lim_{j\to\infty} Q_j(g) = \int_{\mathbb{R}} g(x)\varphi_1(x)dx.$$

- $\Delta_j := Q_j - Q_{j-1}$, $Q_{-1} := 0$. Hence, $\lim_{j\to\infty} \left|\Delta_j g\right| = 0$.

Definition

Given $\mathcal{I} \subset \mathbb{N}_0^d$ (*admissible*), define

$$Q_{\mathcal{I}}(f) := \sum_{\alpha\in\mathcal{I}} \Delta_{\alpha_1} \otimes \cdots \otimes \Delta_{\alpha_d} f.$$

- Example: $\mathcal{I} = \left\{ \alpha \in \mathbb{N}_0^d \,\middle|\, |\alpha| \le L \right\}$ for some L.

$$\int_{\mathbb{R}^d} f(x)\varphi_d(x)dx = ?, \quad \varphi_d \ldots \text{ standard normal density}$$

- Q_j, $j \in \mathbb{N}$, sequence of 1-dim. (Gaussian) quadrature formulas:

$$Q_j(g) = \sum_{\ell=1}^{N_j} v_\ell^{(j)} g(x_\ell^{(j)}), \quad \lim_{j\to\infty} Q_j(g) = \int_{\mathbb{R}} g(x)\varphi_1(x)dx.$$

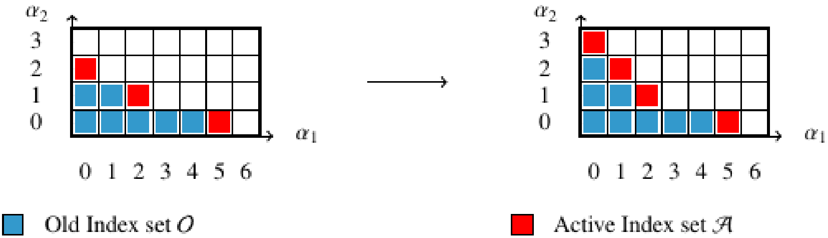
- $\Delta_j := Q_j - Q_{j-1}$, $Q_{-1} := 0$. Hence, $\lim_{j\to\infty} \left|\Delta_j g\right| = 0$.

Definition

Given $\mathcal{I} \subset \mathbb{N}_0^d$ (*admissible*), define

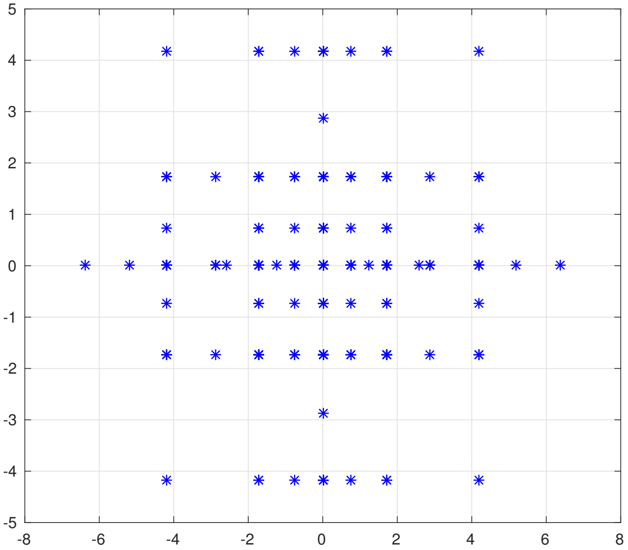
$$Q_{\mathcal{I}}(f) := \sum_{\alpha\in\mathcal{I}} \Delta_{\alpha_1} \otimes \cdots \otimes \Delta_{\alpha_d} f.$$

- Example: $\mathcal{I} = \left\{ \alpha \in \mathbb{N}_0^d \;\middle|\; |\alpha| \le L \right\}$ for some L.



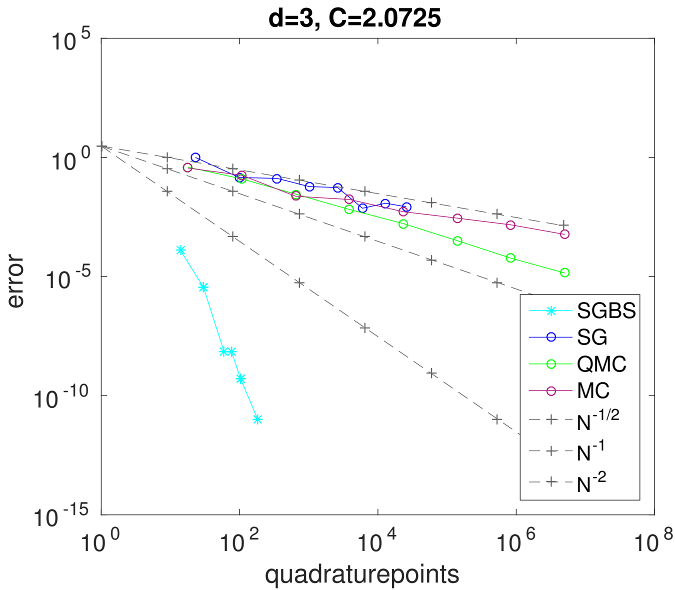
- Initial index set $\mathcal{I} = \{ (0, \ldots, 0) \}$
- Candidate indices: α neighboring (along all axes) $\mathcal{I}$
- For each candidate α evaluate local error estimator g_α, e.g., $g_\alpha = |\Delta_\alpha f|$
- Add candidate α with largest error provided that $g_\alpha \geq \mathrm{TOL}$
- Use 1-dimensional Genz–Keister or Gauss–Hermite quadrature formulae as building blocks.

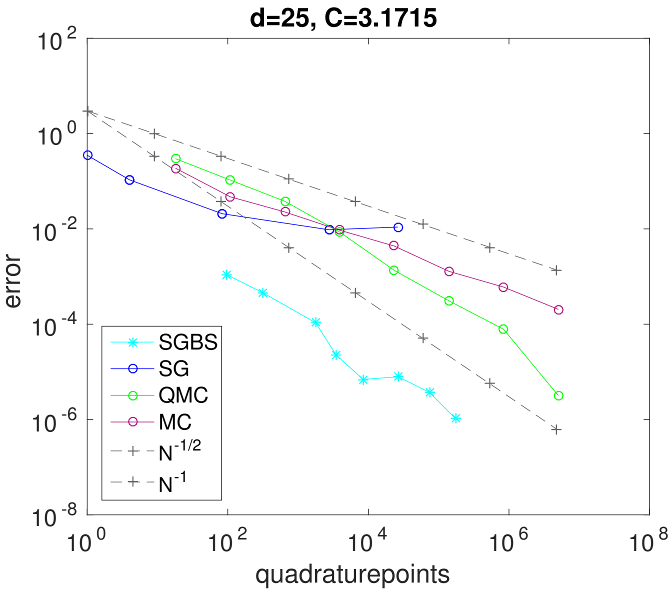
An example of a sparse grid



Outline

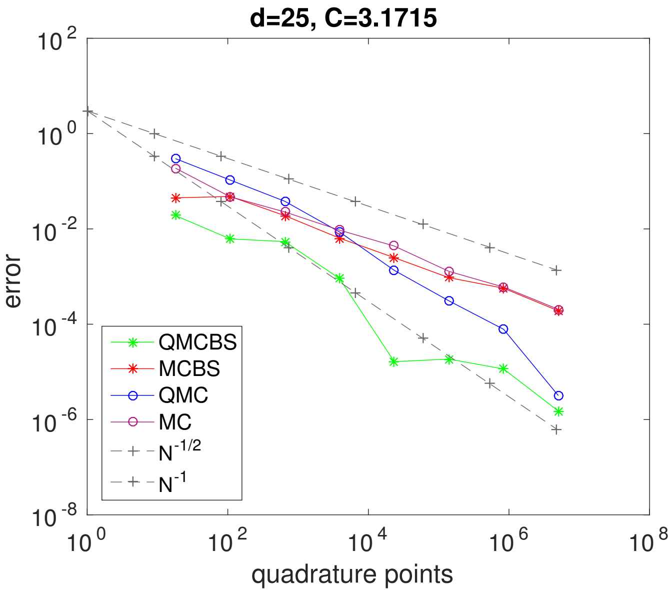
Smoothed adaptive sparse grids, $d = 3$




d=25, C=3.1715
error
quadraturepoints
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^0
10^2
10^4
10^6
10^8
SGBS
SG
QMC
MC
N^-1/2
N^-1

	time	error	points
	SGBS		
$d = 3$	0.0057	4.9×10^{-10}	104
$d = 8$	0.3675	1.81×10^{-9}	2.46×10^{4}
$d = 25$	5.4283	1.04×10^{-6}	1.74×10^{5}
	QMC		
$d = 3$	0.0016	1.25×10^{-1}	108
$d = 8$	0.0161	5.39×10^{-3}	2.33×10^{4}
$d = 25$	0.2406	6.18×10^{-4}	1.40×10^{5}
	MC		
$d = 3$	0.0013	1.77×10^{-1}	108
$d = 8$	0.0135	1.38×10^{-2}	2.33×10^{4}
$d = 25$	0.2188	1.29×10^{-3}	1.40×10^{5}

Example for $d = 25$: $\frac{\text{SGBS-error}}{\text{QMC-error}} \approx \frac{1}{600}$, $\frac{\text{SGBS-time}}{\text{QMC-time}} \approx 23$.


d=25, C=3.1715
error
quadrature points
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^0
10^2
10^4
10^6
10^8
QMCBS
MCBS
QMC
MC
N^-1/2
N^-1

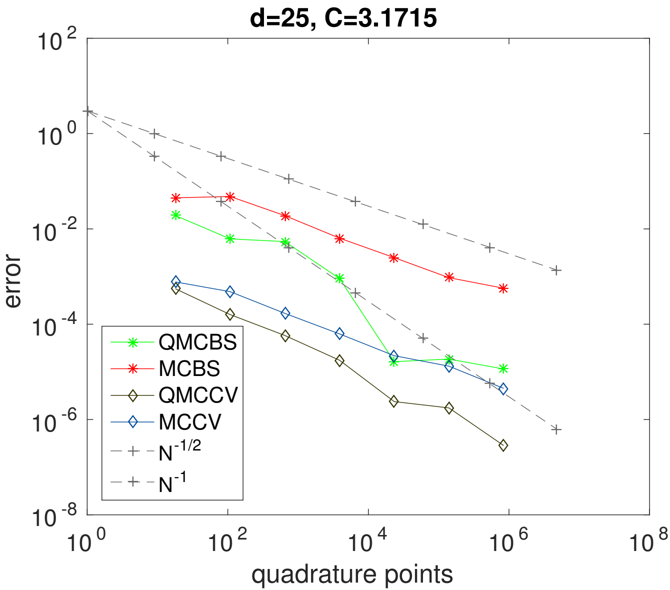
Smoothed adaptive sparse grids as variance reduction

- *Control variates:* $E[f(X)] = E[f(X) - g(X)] + E[g(X)]$
- Assumption: $\mathrm{var}(f(X) - g(X)) < \mathrm{var}(f(X))$, $E[g(X)]$ known
- Choose $g(x)$ as interpolation of f based on sparse grid points
- Hence, $E[g(X)] = Q_{\mathcal{I}} g = Q_{\mathcal{I}} f$
- Improve the integration error by applying (Q)MC on $f(X) - g(X)$
- Note: theoretical justification for control variates with QMC is unclear, but it often works in practice!

Smoothed adaptive sparse grids as variance reduction

- *Control variates:* $E[f(X)] = E[f(X) - g(X)] + E[g(X)]$
- Assumption: $\mathrm{var}(f(X) - g(X)) < \mathrm{var}(f(X))$, $E[g(X)]$ known
- Choose $g(x)$ as interpolation of f based on sparse grid points
- Hence, $E[g(X)] = Q_{\mathcal{I}} g = Q_{\mathcal{I}} f$
- Improve the integration error by applying (Q)MC on $f(X) - g(X)$
- Note: theoretical justification for control variates with QMC is unclear, but it often works in practice!

Smoothed adaptive sparse grids as variance reduction, $d = 25$



Outlook

- Construct *optimal* sparse grid (i.e., replace error estimate by precise error expansions)
- Limitations: e.g., best of call option with payoff $\left(\max_{i=1,\dots,d} c_i S_T^i - K\right)^+$. Smoothing only removes $(\cdot)^+$, not the $\max$.
- Apply for models based on stochastic differential equations:
  - Inherent smoothing available in stochastic volatility models:

$$dS_t = \sqrt{v_t} S_t (\rho dW_t + \sqrt{1-\rho^2} dW_t^\top), \quad dv_t = \mu(v_t)dt + \xi(v_t)dW_t$$

$$E[(S_T - K)^+] = E\Big[C_{BS}\Big(\text{spot} = S_0 e^{\rho \int_0^T \sqrt{v_t} dW_t - \frac{\rho^2}{2} \int_0^T v_t dt},$$
$$\text{strike} = K, \text{vol} = \frac{1-\rho^2}{2} \int_0^T v_t dt\Big)\Big]$$

  - Use highly efficient 1D quadrature coupled with regression when no explicit formulas available

Given a random variable F, we try to compute $E[F]$. (Idea: F is solution of random PDE or SDE)

- $F^\alpha \approx F$, $\alpha \in \mathbb{N}^d$ ("discretization")
- Apply quadrature Q^β and obtain $F_{\alpha,\beta} := Q^\beta(F^\alpha) \approx E[F]$ based on polynomial approximation, $\beta \in \mathbb{N}^l$

Multi-index stochastic collocation (MISC) [Haji-Ali, Nobile, Tamellini, Tempone '16]

$$\mathcal{M}_{\mathcal{I}}(F) := \sum_{(\alpha,\beta)\in\mathcal{I}} \Delta F_{\alpha,\beta}, \quad \mathcal{I} \subset \mathbb{N}^{d\cdot l}$$

- "Sparsify" grid jointly in the discretization and the integration space
- Fast library by R. Tempone's group available, leads to similar performance as reported above in the Black-Scholes basket case

References

- N. Achtsis, R. Cools, D. Nuyens: *Conditional sampling for barrier option pricing under the LT method*, SIFIN, 2013.
- C. B., P. Laurence: *Asymptotics beats Monte Carlo: The case of correlated local vol baskets*, CPAM, 2014.
- H.J. Bungartz, M. Griebel: *Sparse grids*, Acta Numerica, 2004.
- R. Cools: *An encyclopaedia of cubature formulas*, J. Comp., 2003.
- A. Genz, B. D. Keister: *Fully symmetric interpolatory rules for multiple integrals over infinite regions with Gaussian weights*, J. Com. Appl. Math., 1996.
- M. Griebel, F. Kuo, I. Sloan: *The smoothing effect of integration in $\mathbb{R}^d$ and the ANOVA decomposition*, Math. Comp., 2013.
- A.-L. Haji-Ali, F. Nobile, L. Tamellini, R. Tempone: *Multi-index stoch. collocation for random PDEs*, C. M. A. Mech. Eng., 2016.
- M. Holtz: *Sparse grid quadrature in high dimensions with applications in finance and insurance*, Springer, 2014.