Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Rough paths and rough partial differential equations

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Outline

Standard ordinary differential equation

$$\dot{y}_t = V(y_t), \quad y_0 = \xi \in \mathbb{R}^d, \quad t \in [0,1]$$

$V : \mathbb{R}^d \to \mathbb{R}^d$ smooth

Controlled differential equation

$$dy_t = V(y_t)dx_t, \quad y_0 = \xi \in \mathbb{R}^d, \quad t \in [0,1]$$

- $V : \mathbb{R}^d \to \mathbb{R}^{d \times e}$ smooth
- x_t path taking values in $\mathbb{R}^e$
- x_t may contain component t, i.e., includes

$$dy_t = V_0(y_t)dt + V(y_t)dx_t$$

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$$\dot{y}_t = V(y_t)\dot{x}_t$$

- $x_t = W_t(\omega)$ is a path of a Brownian motion, i.e., $y_t = y_t(\omega)$ is pathwise solution of the stochastic differential equation

$$dy_t(\omega) = V(y_t(\omega))dW_t(\omega)$$

(Ito, Stratonovich or some other sense?)

- $x_t = Z_t(\omega)$ for some other stochastic process, such as fractional Brownian motion, $y_t = y_t(\omega)$ is pathwise solution of the corresponding stochastic differential equation

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Integral form

$$dy_t = V(y_t)dx_t, \quad y_0 = \xi \in \mathbb{R}^d, \quad t \in [0,1]$$

Assume that x_t is not smooth, say

$$x \in C^\alpha([0,1];\mathbb{R}^e) := \left\{ x \in C\left([0,1];\mathbb{R}^e\right) \,\middle|\, \sup_{s \neq t} \frac{|x_s - x_t|}{|s-t|^\alpha} =: \|x\|_\alpha < \infty \right\}, \ \alpha < 1$$

- While $\dot{x}$ does not "easily" make sense, maybe the integral form does:

$$y_t = \xi + \int_0^t V(y_s)dx_s, \quad t \in [0,1]$$

- Notice: If $x \in C^\alpha$, then generically $y \in C^\alpha$ (and no better), as well.
- Need to make sense of expressions of the form

$$\int_0^t y_s dx_s, \quad x, y \in C^\alpha([0,1])$$

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Young integral

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Recall the Riemann-Stieltjes integral:

$$\int_0^1 y_s dx_s := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} y_s \underbrace{(x_t - x_s)}_{=: x_{s,t}} \qquad (*)$$

$\mathcal{P}$ a finite *partition* of $[0,1]$

Theorem (Young 1936)

(a) Let $y \in C^\beta([0,1];\mathbb{R})$, $x \in C^\alpha([0,1];\mathbb{R})$ with $0 < \alpha, \beta < 1$ and $\alpha + \beta > 1$. Then $(*)$ converges and the resulting bi-linear map $(x,y) \mapsto \int_0^1 y_s dx_s$ is continuous, i.e., $\left|\int_0^1 y_s dx_s\right| \le C_{\alpha+\beta}(|y_0|) \|y\|_\beta \|x\|_\alpha$.

(b) Let $\alpha + \beta \le 1$. Then there are $y \in C^\beta([0,1];\mathbb{R})$, $x \in C^\alpha([0,1];\mathbb{R})$ such that $(*)$ does not converge, i.e., such that different sequences of partitions yield different limits (or none at all).

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$$dy_t = V(y_t)dx_t, \quad y_0 = \xi \in \mathbb{R}^d, \quad t \in [0,1]$$

Let $x \in C^\alpha([0,1];\mathbb{R}^e)$, $\alpha > \frac{1}{2}$ and $V \in C_b^2(\mathbb{R}^d;\mathbb{R}^{d\times e})$. Then the usual Picard iteration scheme converges and the controlled differential equation has a unique solution.

Example

Let $0 < H < 1$. The *fractional Brownian motion* with Hurst index H is the Gaussian process (on $[0,1]$) with $W_0^H = 0$, $E\left[W_t^H\right] = 0$ and

$$E\left[W_t^H W_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- fBm with $H = \frac{1}{2}$ is standard Brownian motion;
- Paths of W^H are a.s. α-Hölder for any $\alpha < H$ (but no $\alpha \geq H$).

Hence, we can solve fractional SDEs for $H > \frac{1}{2}$.

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Idea

- Choose sequence x^n of smooth paths converging to x
- Assume that corresponding solutions y^n converge to some path $y \in C^\alpha\left([0,1];\mathbb{R}^d\right)$
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A counter-example

$$x_t^n = \left(\sin(n^2 t)/n\,,\ \cos(n^2 t)/n\right), \quad t \in [0, 2\pi]$$

Consider the area function

$$z_t^n$$

Even though $x^n \to 0$ in $\|\cdot\|_\infty$, we have $z_t^n \to -\frac{1}{2}t$.

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$$z_t^n = -\frac{1}{2}\left(\int_0^t \sin\left(n^2 s\right)^2 ds + \int_0^t \cos\left(n^2 s\right)^2 ds\right)$$

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Relevance for controlled differential equations: choose

$$V(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2}y_2 & -\frac{1}{2}y_1 \end{pmatrix}, \quad y \in \mathbb{R}^3$$

Then $y_t^n := \left(x_t^{n,1}, x_t^{n,2}, z_t^n\right)$ solves

$$dy_t^n = V(y_t^n)dx_t^n, \quad y_0 = (0, 1/n, 0)\,.$$

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Remark

- *The example is not just an instance of "poor choice of norm": replacing $\|\cdot\|_\infty$ by any other reasonable norm is vulnerable to the same type of example.*
- *"Curing this example will cure all other counter-examples."*
- *Does not work in dimension $e = 1$ (Doss–Sussmann transformation.)*

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Suppose you want to cover the case $x_t = W_t(\omega)$, a standard Brownian motion.

Brownian motion is a *martingale*: i.e., the increments are orthogonal (in $L^2(\Omega)$) to the past: for bounded f, we have

$$Z = f\left((W_u)_{0\le u\le s}\right) \Rightarrow E\left[ZW_{s,t}\right] = 0 \text{ for } 0 < s < t.$$

This strong geometric condition allows to define

$$\int_0^t Z_s dW_s = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} Z_u W_{u,v} \text{ in probability,}$$

provided that Z is adapted (i.e., $\forall s:\ Z_s$ is $\sigma\left((W_u)_{0\le u\le s}\right)$-measurable) and square integrable w.r.t. $dt \otimes P$.

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Let Φ denote the solution map $x \mapsto y$ for x smooth (discontinuous).

Rough path principle

- By continuity of Ψ, can define y as limit of smooth solutions
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- Rough path theory does not help with actual construction of $\mathbf{x}$.
- Use Ito/Stratonovich stochastic integral in case of Brownian motion. No pathwise construction of $\mathbf{x} = \mathbf{x}(\omega)$, but pathwise construction of $y = y(\omega)$ given a path of $\mathbf{x}$.

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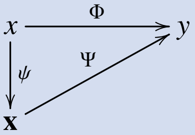

Enhance x to a rough path $\mathbf{x}$, such that the solution map $\Psi : \mathbf{x} \to y$ is *continuous* (in rough path topology).

- By continuity of Ψ, can define y as limit of smooth solutions
- Morally, $\mathbf{x} = \left(x, \int_0^\cdot x_s \otimes dx_s\right)$
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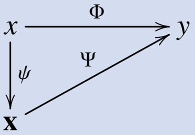

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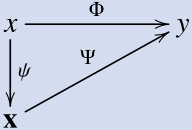

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Outline

Let $x : [0,1] \to \mathbb{R}^e$ be a smooth path, $x_{s,t} := x_t - x_s$ and consider

$$\mathbb{x}_{s,t} := \int_s^t x_{s,u} \otimes dx_u := \left(\int_s^t x^i_{s,u} dx^j_u \right)^e_{i,j=1}$$

How do increments of of $\mathbb{x}$ behave? Let $s < u < t$, then

$$\mathbb{x}_{s,t} = \int_s^t x_{s,v} \otimes dx_v$$

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Theorem (Chen's theorem)

$$\mathbb{x}_{s,t} - \mathbb{x}_{s,u} - \mathbb{x}_{u,t} = x_{s,u} \otimes x_{u,t}$$

Remark

Note $\mathbb{x}_{s,t} \to \mathbb{x}_{s,t} + f_t - f_s$ leaves Chen's relation invariant.

Chen's relation

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Let $\frac{1}{3} < \alpha \le \frac{1}{2}$. The space of α-Hölder rough paths $\mathscr{C}^\alpha([0,1],\mathbb{R}^e)$ is the set of pairs $\mathbf{x} = (x, \mathbb{x})$, $x : [0,1] \to \mathbb{R}^e$, $\mathbb{x} : [0,1]^2 \to \mathbb{R}^e \otimes \mathbb{R}^e$ such that

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Remark

- *$\mathscr{C}^\alpha$ is not a linear space, but a closed subset of a Banach space.*
- *For $\alpha \le \frac{1}{3}$, need to add iterated integrals of order up to $\lfloor \frac{1}{\alpha} \rfloor$.*
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Notice that $\|x\|_\alpha + \|\mathbb{x}\|_{2\alpha}$ is not homogeneous under the natural dilatation $\lambda \mapsto \left(\lambda x, \lambda^2 \mathbb{x}\right)$.

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The *homogeneous rough path (semi-) norm* is defined by

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$\mathscr{C}^\alpha\left([0,1];\mathbb{R}^e\right)$ is a complete metric space under ϱ_α.

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A rough path $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$ is called geometric—symbolically, $\mathbf{x} \in \mathscr{C}^\alpha_g([0,1];\mathbb{R}^e)$)—iff

$$\operatorname{sym}(\mathbb{x}) := \frac{1}{2}\left(\mathbb{x}^{i,j}_{\cdot,\cdot} + \mathbb{x}^{j,i}_{\cdot,\cdot}\right)^e_{i,j=1} = \frac{1}{2}\left(x_{\cdot,\cdot} \otimes x_{\cdot,\cdot}\right).$$

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For a smooth path x define $I_2(x) := (x, \mathbb{x})$ with $\mathbb{x}^{i,j}_{s,t} := \int_s^t x^i_{s,u} dx^j_u$. Then $\mathscr{C}^\alpha_g$ contains the closure of the subset of $\mathscr{C}^\alpha$ obtained as image of smooth paths under I_2.

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Given an e-dimensional Brownian motion $B = B(\omega)$.

- Is there a rough path $\mathbf{B} = (B, \mathbb{B})$?
- Is it unique, which properties does it have?

$$\mathbb{B}^{\text{Ito}}_{s,t} := \int_s^t B_{s,u} \otimes dB_u = \lim_{\mathcal{P}\subset[s,t],\, |\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} B_{s,u} \otimes B_{u,v}$$

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For any $\alpha < \frac{1}{2}$ we have

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Outline

Integration of 1-forms – motivation

For $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$ and $f : \mathbb{R}^e \to \mathbb{R}^{d\times e}$, we want to construct

$$\int_0^1 f(x_s)d\mathbf{x}_s \quad \text{or even} \quad z = \int_0^\cdot f(x_s)d\mathbf{x}_s \quad \text{or even} \quad \mathbf{z} = \int_0^\cdot f(x_s)d\mathbf{x}_s$$

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Idea of Riemann-Stieltjes integral for rough x:

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$$\Rightarrow \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} f(x_s)x_{s,t} \text{ does not exist in general}$$

For $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$ and $f : \mathbb{R}^e \to \mathbb{R}^{d\times e}$, we want to construct

$$\int_0^1 f(x_s)d\mathbf{x}_s \quad \text{or even} \quad z = \int_0^\cdot f(x_s)d\mathbf{x}_s \quad \text{or even} \quad \mathbf{z} = \int_0^\cdot f(x_s)d\mathbf{x}_s$$

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Theorem (Lyons)

Let $\alpha > \frac{1}{3}$ and $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $f \in C_b^2\left(\mathbb{R}^e, \mathbb{R}^{d\times e}\right)$. Then the rough integral

$$\int_0^1 f(x_s)d\mathbf{x}_s := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \left(f(x_s)x_{s,t} + Df(x_s)\mathbb{x}_{s,t}\right)$$

exists and satisfies

$$\left|\int_s^t f(x_u)d\mathbf{x}_u - f(x_s)x_{s,t} - Df(x_s)\mathbb{x}_{s,t}\right| \leq C_\alpha \|f\|_{C_b^2}\left(\|x\|_\alpha^3 + \|x\|_\alpha \|\mathbb{x}\|_{2\alpha}\right)|t-s|^{3\alpha}.$$

Moreover, $\int_0^\cdot f(x_u)d\mathbf{x}_u$ is α-Hölder continuous with

$$\left\|\int_0^\cdot f(x_u)d\mathbf{x}_u\right\|_\alpha \leq C_\alpha \|f\|_{C_b^2} \max\left(|||\mathbf{x}|||_\alpha, |||\mathbf{x}|||_\alpha^{1/\alpha}\right).$$

First some notation:

$$\begin{aligned} y_s &:= f(x_s), \\ y'_s &:= Df(x_s), \\ \Xi_{s,t} &:= y_s x_{s,t} + y'_s \mathbb{X}_{s,t} \\ \delta\Xi_{s,u,t} &:= \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t} \end{aligned}$$

We prove convergence

$$\lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \Xi_{s,t} =: \lim_{|\mathcal{P}|\to 0} \int_{\mathcal{P}} \Xi =: \int \Xi,$$

i.e., the limit does not depend on the sequence of partitions.

Lemma

$$\|\Xi\|_{\alpha,3\alpha} := \|\Xi\|_{\alpha} + \|\delta\Xi\|_{3\alpha} < \infty \textit{ with } \|\delta\Xi\|_{\beta} := \sup_{s<u<t} \left|\delta\Xi_{s,u,t}\right| / |t-s|^{\beta}.$$

Proof.

- Clearly, $\|y\|_{\alpha} \le \|Df\|_{\infty} \|x\|_{\alpha} < \infty, \quad \|y'\|_{\alpha} \le \left\|D^2 f\right\|_{\infty} \|x\|_{\alpha} < \infty$.
- Consider $R_{s,t} := y_{s,t} - y'_s x_{s,t}$ and $g(\xi) := f(x_s + \xi x_{s,t})$, $\xi \in [0,1]$.
- By Taylor's formula, there is $\xi \in [0,1]$ s.t.

$$R_{s,t} = g(1) - g(0) - g'(0) = \frac{1}{2} g''(\xi) = \frac{1}{2} D^2 f(x_s + \xi x_{s,t}) \cdot (x_{s,t}, x_{s,t})$$

- Using Chen's relation $\mathbb{x}_{s,t} = \mathbb{x}_{s,u} + \mathbb{x}_{u,t} + x_{s,u} \otimes x_{u,t}$, we have

$$\begin{aligned}\delta\Xi_{s,u,t} &= (y_s x_{s,t} + y'_s \mathbb{x}_{s,t}) - (y_s x_{s,u} + y'_s \mathbb{x}_{s,u}) - (y_u x_{u,t} + y'_u \mathbb{x}_{u,t}) \\ &= -y_{s,u} x_{u,t} + y'_s x_{s,u} \otimes x_{u,t} - (y'_u - y'_s) \mathbb{x}_{u,t} \\ &= -R_{s,u} \otimes x_{u,t} - (y'_u - y'_s) \mathbb{x}_{u,t}\end{aligned}$$

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\delta\Xi_{s,u,t} &= (y_s x_{s,t} + y'_s \mathbb{x}_{s,t}) - (y_s x_{s,u} + y'_s \mathbb{x}_{s,u}) - (y_u x_{u,t} + y'_u \mathbb{x}_{u,t}) \\
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Lemma

$$\sup_{\mathcal{P}\subset[s,t]} \left| \Xi_{s,t} - \int_{\mathcal{P}} \Xi \right| \leq 2^{3\alpha} \, \|\delta\Xi\|_{3\alpha} \, \zeta(3\alpha) \, |t-s|^{3\alpha} \; (*)$$

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Proof.

Indeed, let $\mathcal{P} \subset [s,t]$ with $r := \#\mathcal{P}$. If $r \geq 2$, then

$$\exists u < v < w : \; [u,v],[v,w] \in \mathcal{P} \text{ and } |w-u| \leq \frac{2\,|t-s|}{r-1}.$$

Hence,

$$\left| \int_{\mathcal{P}\setminus\{v\}} \Xi - \int_{\mathcal{P}} \Xi \right| = \left| \delta\Xi_{u,v,w} \right| \leq \|\delta\Xi\|_{3\alpha} \left(\frac{2\,|t-s|}{r-1} \right)^{3\alpha}.$$

Iterating the procedure until $\#\mathcal{P} = 1$ gives the assertion. □

Integration of 1-forms – Existence III

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$$\lim_{\epsilon\searrow 0} \sup_{\max(|\mathcal{P}|,|\mathcal{P}'|)<\epsilon} \left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| = 0$$

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Proof.

W.l.o.g., $\mathcal{P}' \subset \mathcal{P}$. By definition of $\int \Xi$ and $(*)$, we get

$$\int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi = \sum_{[u,v]\in\mathcal{P}} \left(\Xi_{u,v} - \int_{\mathcal{P}'\cap[u,v]} \Xi \right)$$

$$\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \leq 2^{3\alpha} \zeta(3\alpha) \, \|\delta\Xi\|_{3\alpha} \sum_{[u,v]\in\mathcal{P}} |v-u|^{3\alpha} = \mathcal{O}\left(|\mathcal{P}|^{3\alpha-1}\right) = \mathcal{O}(\epsilon^{3\alpha-1}).$$

- $\int_0^t V(x_s)d\mathbf{x}_s$ ✔
- $\int_0^t V(y_s)d\mathbf{x}_s$ **?**

Given $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $\mathbf{y} \in \mathscr{C}^\alpha\left([0,1];\mathbb{R}^d\right)$ with $\alpha \leq \frac{1}{2}$, it is generally not possible to construct

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unless there is $\mathbf{z} \in \mathscr{C}^\alpha\left([0,1];\mathbb{R}^{e+d}\right)$ with $z = (x,y)$—and the result will depend on the choice of $\mathbf{z}$.

- Picard iteration for $dy_s = V(y_s)dx_s$, $y_0 = \xi$:

1. $y^{(0)} \equiv \xi$, then $y^{(1)} := \xi + \int_0^{\cdot} V(y_s^{(0)})d\mathbf{x}_s$ defined ✔
2. $y^{(1)} \equiv \xi + V(\xi)x$, then $y^{(2)} := \xi + \int_0^{\cdot} V(y_s^{(1)})d\mathbf{x}_s$ defined ✔
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Outline

Definition

Given $x \in C^\alpha([0,1];\mathbb{R}^e)$, $y \in C^\alpha\left([0,1];\mathbb{R}^d\right)$ is called controlled by x, iff there is $y' \in C^\alpha\left([0,1];\mathbb{R}^{d\times e}\right)$ – $\mathbb{R}^{d\times e} = \mathcal{L}(\mathbb{R}^e,\mathbb{R}^d)$ – s.t.

$$R_{s,t} := y_{s,t} - y'_s x_{s,t}$$

satisfies $\|R\|_{2\alpha} < \infty$. We write $(y,y') \in \mathcal{D}_x^{2\alpha}\left([0,1];\mathbb{R}^d\right)$.

Example

If $f \in C_b^2\left(\mathbb{R}^e;\mathbb{R}^d\right)$, $y := f(x)$, $y' := Df(x)$, then $(y,y') \in \mathcal{D}_x^{2\alpha}\left([0,1;\mathbb{R}^d\right)$.

Remark

$\mathcal{D}_x^{2\alpha}$ *is a Banach space under* $(y,y') \mapsto |y_0| + \left|y'_0\right| + \|(y,y')\|_{x,2\alpha}$ *with*

$$\left\|(y,y')\right\|_{x,2\alpha} := \left\|y'\right\|_\alpha + \|R\|_{2\alpha}.$$

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Theorem (Gubinelli)

Let $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $(y,y') \in \mathscr{D}_x^{2\alpha}\left([0,1],\mathbb{R}^{d\times e}\right)$.

a)The integral

$$\int_0^1 y_s d\mathbf{x}_s := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} (y_s x_{s,t} + y'_s \mathbb{x}_{s,t})$$

exists and satisfies

$$\left|\int_s^t y_u d\mathbf{x}_u - y_s x_{s,t} - y'_s \mathbb{x}_{s,t}\right| \le C_\alpha \left(\|x\|_\alpha \|R\|_{2\alpha} + \|\mathbb{x}\|_{2\alpha} \left\|y'\right\|_\alpha\right) |t-s|^{3\alpha}.$$

b) Set $(z,z') := \left(\int_0^\cdot y_s d\mathbf{x}_s, y\right)$*. Then* $(z,z') \in \mathscr{D}_x^{2\alpha}\left([0,1];\mathbb{R}^d\right)$ *and* $(y,y') \mapsto (z,z')$ *is a continuous linear map with*

$$\left\|(z,z')\right\|_{x,2\alpha} \le \|y\|_\alpha + \left\|y'\right\|_\infty \|\mathbb{x}\|_{2\alpha} + C_\alpha \left(\|x\|_\alpha \left\|R^y\right\|_{2\alpha} + \|\mathbb{x}\|_{2\alpha} \left\|y'\right\|_\alpha\right).$$

Controlled rough paths are rough paths: Given $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $(y, y') \in \mathscr{D}_x^{2\alpha}\left([0,1];\mathbb{R}^d\right)$, then we can define $\mathbb{y} \in \mathbb{R}^{d\times d}$ by

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s.t., $\mathbf{y} = (y, \mathbb{y}) \in \mathscr{C}^\alpha\left([0,1];\mathbb{R}^d\right)$. Here, $y' \otimes y' \in \mathcal{L}(\mathbb{R}^{e\times e}, \mathbb{R}^{d\times d})$, $y' \otimes y'(a \otimes b) = y'(a) \otimes y'(b)$.

Composition with regular functions: For $\mathbf{x}$, (y, y') as before, let $\varphi \in C_b^2(\mathbb{R}^d;\mathbb{R}^n)$ and define

$$z_t := \varphi(y_t), \quad z'_t := D\varphi(y_t) \otimes y'_t.$$

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$$\|z\|_\alpha \le \|\varphi\|_{C_b^2} \|y\|_\alpha$$

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$$\left\|z'\right\|_\alpha \le \|D\varphi \circ y\|_\infty \left\|y'\right\|_\alpha + \left\|y'\right\|_\infty \|D\varphi \circ y\|_\alpha$$

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$$z_{s,t} - z'_s x_{s,t} = \frac{1}{2} D^2\varphi\left(y_s + \xi y_{s,t}\right)\left(y_{s,t}, y_{s,t}\right) + D\varphi(y_s)\left(y_{s,t} - y'_s x_{s,t}\right)$$

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$$\left\| z_{s,t} - z'_s x_{s,t} \right\|_{2\alpha} \le \frac{1}{2} \|\varphi\|_{C_b^2} \|y\|_\alpha^2 + \|\varphi\|_{C_b^2} \left\| y_{\cdot,\cdot} - y' x_{\cdot,\cdot} \right\|_{2\alpha}$$

The choice of $\mathbb{x}$ matters

Let $\mathbf{x} = (x, \mathbb{x}) \in \mathscr{C}^{\alpha}([0,1];\mathbb{R}^{e})$ and $f \in C^{2\alpha}([0,1];\mathbb{R}^{e\times e})$. Then

$$\overline{\mathbf{x}} = (\overline{x}, \overline{\mathbb{x}}) \in \mathscr{C}^{\alpha}([0,1],\mathbb{R}^{e}), \quad \overline{x} := x, \quad \overline{\mathbb{x}}_{s,t} := \mathbb{x}_{s,t} + f(t) - f(s).$$

As $\mathcal{D}_{x}^{2\alpha} = \mathcal{D}_{\overline{x}}^{2\alpha}$, we may integrate $(y, y') \in \mathcal{D}_{x}^{2\alpha}\left([0,1];\mathbb{R}^{d\times e}\right)$ against both.

$$\begin{aligned}\int_0^1 y_s d\overline{\mathbf{x}}_s &= \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \left(y_s x_{s,t} + y'_s\left(\mathbb{x}_{s,t} + f(t) - f(s)\right)\right) \\ &= \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \left(y_s x_{s,t} + y'_s \mathbb{x}_{s,t}\right) + \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} y'_s (f(t) - f(s)) \\ &= \int_0^1 y_s d\mathbf{x}_s + \int_0^1 y'_s df(s)\end{aligned}$$

As $\alpha + 2\alpha > 1$, $\int y' df$ is defined in the Young sense.

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Outline

Let $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $V : \mathbb{R}^d \to \mathbb{R}^{d\times e}$ smooth, consider

$$dy_s = V(y_s)dx_s, \quad y_0 = \xi \in \mathbb{R}^d$$

1 $y^{(0)} \equiv \xi$, then $y^{(1)} := \xi + \int_0^\cdot V(y_s^{(0)})d\mathbf{x}_s$ defined ✔
Moreover, $(y^{(1)}, V(y^{(0)})) \in \mathscr{D}_x^{2\alpha}\left([0,1];\mathbb{R}^d\right)$.

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Theorem (Lyons; Gubinelli)

Given $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $\frac{1}{3} < \alpha < \frac{1}{2}$, $V \in C_b^3\left(\mathbb{R}^d;\mathbb{R}^{d\times e}\right)$, $\xi \in \mathbb{R}^d$. *Then there is a unique* $(y, y') \in \mathscr{D}_x^{2\alpha}\left([0,1];\mathbb{R}^d\right)$ *such that*

$$\forall t \in [0,1]: \; y_t = \xi + \int_0^t V(y_s) d\mathbf{x}_s,$$

with $y' = V(y)$.

- If $V \in C^3$, obtain a local solution.
- Existence requires $V \in C^\gamma$ for some $\gamma > \frac{1}{\alpha} - 1$ — i.e., V is $\lfloor\gamma\rfloor$-differentiable with $\lfloor\gamma\rfloor$-derivative in $C^{\gamma-\lfloor\gamma\rfloor}$.
- Uniqueness requires $V \in C^\gamma$ for some $\gamma \geq \frac{1}{\alpha}$.
- For the smooth case "$\alpha = 1$", this essentially recovers standard results from ODE theory.

Theorem (Lyons; Gubinelli)

Given $\mathbf{x} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $\frac{1}{3} < \alpha < \frac{1}{2}$, $V \in C_b^3\left(\mathbb{R}^d;\mathbb{R}^{d\times e}\right)$, $\xi \in \mathbb{R}^d$. *Then there is a unique* $(y, y') \in \mathcal{D}_x^{2\alpha}\left([0,1];\mathbb{R}^d\right)$ *such that*

$$\forall t \in [0,1] : \; y_t = \xi + \int_0^t V(y_s) d\mathbf{x}_s,$$

with $y' = V(y)$.

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- Existence requires $V \in C^\gamma$ for some $\gamma > \frac{1}{\alpha} - 1$ — i.e., V is $\lfloor\gamma\rfloor$-differentiable with $\lfloor\gamma\rfloor$-derivative in $C^{\gamma - \lfloor\gamma\rfloor}$.
- Uniqueness requires $V \in C^\gamma$ for some $\gamma \geq \frac{1}{\alpha}$.
- For the smooth case "$\alpha = 1$", this essentially recovers standard results from ODE theory.

- Given $(y, y') \in \mathcal{D}_x^{2\alpha}$, $T \le 1$, we have $(z, z') := (V(y), DV(y)y') \in \mathcal{D}_x^{2\alpha}$ and we can define

$$\mathcal{M}_T : \mathcal{D}_x^{2\alpha}([0,T];\mathbb{R}^d) \to \mathcal{D}_x^{2\alpha}([0,T];\mathbb{R}^d), \quad (y,y') \mapsto \left(\xi + \int_0^\cdot z_s d\mathbf{x}_s, z\right).$$

- For T small enough, one can show that the closed subset

$$\mathcal{B}_T := \left\{ (y,y') \in \mathcal{D}_x^{2\alpha}\left([0,T];\mathbb{R}^d\right) \,\middle|\, y_0 = \xi, y_0' = V(\xi), \left\|(y,y')\right\|_{x,2\alpha} \le 1 \right\}$$

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is invariant under $\mathcal{M}_T$.

- For T small enough, $\mathcal{M}_T$ is a contraction on $\mathcal{B}_T$, i.e., for $(y, y'), (\widetilde{y}, \widetilde{y}') \in \mathcal{B}_T$:

$$\left\| \mathcal{M}_T(y, y') - \mathcal{M}_T(\widetilde{y}, \widetilde{y}') \right\|_{x,2\alpha} \leq \frac{1}{2} \left\| (y - \widetilde{y}, y' - \widetilde{y}') \right\|_{x,2\alpha}.$$

Need to estimate $V(y_s) - V(\widetilde{y}_s)$ by $y_s - \widetilde{y}_s$, but in rough path sense, i.e.,

$$\left\| (V(y) - V(\widetilde{y}), (V(y) - V(\widetilde{y}))') \right\|_{x,2\alpha} \leq \text{const} \left\| (y - \widetilde{y}, y' - \widetilde{y}') \right\|_{x,2\alpha}.$$

Consider

$$V(y) - V(\widetilde{y}) = g(y, \widetilde{y})(y - \widetilde{y}), \quad g(a, b) := \int_0^1 DV(ta + (1 - t)b)dt$$

$g \in C_b^2$ and $\|g\|_{C_b^2} \leq \text{const} \, \|V\|_{C_b^3}$.

$$dy = V(y)d\mathbf{x}, \quad y_0 = \xi \in \mathbb{R}^d, \quad \mathbf{x} \in \mathscr{C}^\alpha\left([0,1];\mathbb{R}^e\right), \ \frac{1}{3} < \alpha < \frac{1}{2}$$

- From $(y, V(y)) \in \mathcal{D}_x^{2\alpha}$, we know that

$$y_{s,t} = V(y_s)x_{s,t} + O\left(|t-s|^{2\alpha}\right).$$

As $2\alpha < 1$, this Euler scheme will not converge.

- From integration of controlled rough paths and the RDE, we know

$$y_{s,t} =$$

Theorem

The Milstein scheme is converging (with rate $3\alpha - 1 - \epsilon$).

- Including iterated integrals of order up to N will give a scheme with rate $(N+1)\alpha - 1 - \epsilon$, provided V is smooth enough.

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Theorem (Lyons)

Let $\mathbf{x}, \widetilde{\mathbf{x}} \in \mathscr{C}^\alpha([0,1];\mathbb{R}^e)$, $\frac{1}{3} < \alpha < \frac{1}{2}$, $\xi, \widetilde{\xi} \in \mathbb{R}^d$ *and* $(y, V(y)), (\widetilde{y}, V(\widetilde{y})) \in \mathcal{D}_x^{2\alpha}\left([0,1];\mathbb{R}^d\right)$ *the unique solutions to*

$$\begin{aligned} dy &= V(y)d\mathbf{x}, \quad y_0 = \xi, \\ d\widetilde{y} &= V(\widetilde{y})d\widetilde{\mathbf{x}}, \quad \widetilde{y}_0 = \widetilde{\xi}. \end{aligned}$$

Let $|||\mathbf{x}|||_\alpha, |||\widetilde{\mathbf{x}}|||_\alpha \leq M < \infty$. *Then there is a constant* $C = C(M, \alpha, \|V\|_{C_b^3})$ *such that*

$$\|y - \widetilde{y}\|_\alpha \leq C\left(\left|\xi - \widetilde{\xi}\right| + \varrho_\alpha(\mathbf{x}, \widetilde{\mathbf{x}})\right).$$

This result can be extended to the full rough path $\mathbf{y}$ and $\widetilde{\mathbf{y}}$.

Outline

Stochastic differential equations

For $V \in C_b^3(\mathbb{R}^d; \mathbb{R}^{d\times e})$, $V_0 : \mathbb{R}^d \to \mathbb{R}^d$ Lipschitz, consider

$$dY_t = V_0(Y_t)dt + V(Y_t)dB_t, \quad Y_0 = \xi.$$

Recall the Ito and Stratonovich Brownian rough paths $\mathbf{B}^{\text{Ito}}$ and $\mathbf{B}^{\text{Strat}}$.

Theorem

a) For any ω such that $\mathbf{B}^{\text{Ito}}(\omega) \in \mathscr{C}^\alpha$, denote by $Y = Y(\omega)$ the unique solution of the RDE

$$dY_t(\omega) = V_0(Y_t(\omega))dt + V(Y_t(\omega))d\mathbf{B}_t^{\text{Ito}}(\omega), \quad Y_0(\omega) = \xi.$$

Then Y is a strong solution of the above Ito SDE.

b) For any ω such that $\mathbf{B}^{\text{Strat}}(\omega) \in \mathscr{C}_g^\alpha$, denote by $Y = Y(\omega)$ the unique solution of the RDE

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B^n ... piece-wise linear approximations of a Brownian motion B

$$dY_t^n = V(Y_t^n)dB_t^n, \quad Y_0 = \xi, \ V \in C_b^3(\mathbb{R}^e, \mathbb{R}^{d\times e}).$$

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Y^n converges a.s. to the Stratonovich solution

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More precisely, we have $\|Y - Y^n\|_\alpha \to 0$ a.s. for $\alpha < \frac{1}{2}$.

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Proof.

- Consider $\mathbb{B}_{s,t}^n := \int_s^t B_{s,u}^n dB_u^n$, show that $\left\|\mathbb{B}^n - \mathbb{B}^{\text{Strat}}\right\|_{2\alpha} \to 0$ a.s.
- Apply the universal limit theorem:

$$\left\|Y - Y^n\right\|_\alpha \leq \text{const}\ \varrho_\alpha\left(\mathbb{B}^n, \mathbb{B}^{\text{Strat}}\right).$$

□

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- Non dyadic pice-wise linear approximations possible, lead to convergence rate $\frac{1}{2} - \alpha - \epsilon$—in C^α. By working in spaces with lower regularity, one can get to $\frac{1}{2} - \epsilon$.
- The result also holds—mutatis mutandis—for fractional Brownian motion with $H > \frac{1}{4}$.

Outline

$$du = F[u]dt + \sum_{i=1}^{d} H_i[u] \circ dW_t^i(\omega), \quad u(0,x) = g(x), \quad x \in \mathbb{R}^n,$$

$$F[u] = F(x, u, Du, D^2u),$$

$$H_i[u] = H_i(x, u, Du), \quad i = 1, \dots, e.$$

We assume

Transport noise: $H_i[u] = \langle \beta_i(x),\ Du \rangle$

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Idea

1. Solve the equation with mollified noise
2. Show that limiting solution only depends on rough path $\mathbf{W}$
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Flow transformation method

$$du = F(x, u, Du, D^2u)dt + \langle \beta(x),\ Du \rangle \circ dW_t, \quad u(0, x) = g(x) \qquad (*)$$

Apply a (W-dependent) transformation turning ($*$) into a deterministic "classical" PDE, provided that W is smooth.

Let $y_t = \varphi_t(\xi)$ denote the flow of the ODE $\dot{y}_t = -\beta(y_t)\dot{W}_t$, $y_0 = \xi \in \mathbb{R}^n$.

Theorem

u is a classical solution of

$$\partial_t u = F(x, u, Du, D^2u) + \langle \beta(x),\ Du \rangle\, \dot{W}$$

if and only if $v(t, x) := u(t, \varphi_t(x))$ is a classical solution to

$$\partial_t v = F^{\varphi}(t, x, v, Dv, D^2v)$$

with

$$F^{\varphi}(t, \varphi_t(x), r, p, X) \equiv F\left(x, r, \left\langle p,\ D\varphi_t^{-1}\right\rangle, \left\langle X,\ D\varphi_t^{-1} \otimes D\varphi_t^{-1}\right\rangle + \left\langle p,\ D^2\varphi_t^{-1}\right\rangle\right).$$

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1. Pick $\mathbf{W} \in \mathscr{C}_g^{\alpha}([0, 1]; \mathbb{R}^e)$ together with a sequence W^ϵ of smooth paths approximating $\mathbf{W}$.
2. By the universal limit theorem for RDEs, we have

$$F^\epsilon := F^{\varphi^\epsilon} \xrightarrow{\epsilon \to 0} F^{\varphi}$$

   φ^ϵ and φ denoting the flows of

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For the third step we need a (deterministic) PDE framework for solutions $u \in \mathcal{U}$ and initial conditions $g \in \mathcal{G}$ such that

(i) For $g^\epsilon \in \mathcal{G}$, the approximate problem

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admits a unique solution $u^\epsilon \in \mathcal{U}$.

(ii) $u^\epsilon \in \mathcal{U}$ solves $(*)$ if and only if $v^\epsilon(t,x) := u(t, \varphi_t^\epsilon(x)) \in \mathcal{U}$ solves

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(iii) When $g^\epsilon \to g \in \mathcal{G}$ and $F^\epsilon \to F^0$—as seen for $F^0 = F^\varphi$—then $v^\epsilon \to v^0$, the unique solution to $\partial_t v^0 = F^0(t, x, v^0, Dv^0 . D^2 v^0)$.

(iv) $v^\epsilon \to v^0$ in $\mathcal{U}$ implies that $u^\epsilon \to u^0$, with $v^0(t,x) = u^0(t, \varphi_t(x))$.

For the third step we need a (deterministic) PDE framework for solutions $u \in \mathcal{U}$ and initial conditions $g \in \mathcal{G}$ such that

(i) For $g^\epsilon \in \mathcal{G}$, the approximate problem

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For our model problem, the concept of viscosity solutions satisfies the requirements for $\mathcal{U} = BC([0,1] \times \mathbb{R}^n)$, $\mathcal{G} = BUC(\mathbb{R}^n)$ provided that

- F is degenerate elliptic and satisfies some technical conditions;
- For all C_b^3-diffeomorphisms φ, comparison holds for F^φ.

Theorem

Given $\frac{1}{3} < \alpha \leq \frac{1}{2}$, $\mathbf{W} \in \mathscr{C}_g^\alpha$ and a $W^\epsilon \in C^1([0,1]; \mathbb{R}^e)$ such that

$$\mathbf{W}^\epsilon := (W^\epsilon, \mathbb{W}^\epsilon) \xrightarrow[\text{in } \mathscr{C}^\alpha]{\epsilon \to 0} \mathbf{W}, \quad \mathbb{W}^\epsilon_{s,t} := \int_0^t W^\epsilon_{s,u} \otimes dW^\epsilon_u.$$

Consider the unique viscosity solution $u^\epsilon \in BC$ to

$$\partial_t u^\epsilon = F(x, u^\epsilon, Du^\epsilon, D^2 u^\epsilon) + \langle \beta(x), Du^\epsilon \rangle \dot{W}^\epsilon, \quad u^\epsilon(0, \cdot) = g.$$

- $\exists u = \lim_{\epsilon \to 0} u^\epsilon \in BC$ *(locally uniformly). u only depends on $\mathbf{W}$.*
- *The transformation v of u is the unique solution of* $\partial_t v = F^\varphi(t, x, v, Dv, D^2 v)$ *in BC, φ being the flow of $dy = -\beta(y) d\mathbf{W}$.*
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Theorem (Lyons '91)

There is no separable Banach space $\mathcal{B} \subset C([0,1])$*, such that*

1. $\bigcap_{0<\alpha<\frac{1}{2}} C^{\alpha}([0,1]) \subset \mathcal{B}$;
2. *the bi-linear map*

$$(f,g) \mapsto \int_0^{\cdot} f(s)\dot{g}(s)ds$$

defined on $C^{\infty}([0,1]) \times C^{\infty}([0,1])$ *extends to a continuous map*

$$\mathcal{B} \times \mathcal{B} \to C([0,1]).$$

▸ Back

Consider $G : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \to \mathbb{R}$ continuous and degenerate elliptic, i.e.,

$$B \geq 0 \Rightarrow G(x, u, p, A) \leq G(x, u, p, A + B).$$

A continuous function u is a viscosity supersolution of

$$-G(x, u, Du, D^2 u) \geq 0$$

iff for every smooth test-function ϕ touching u from below in some point x_0, we have

$$-G(x_0, \phi, D\phi, D^2\phi) \geq 0.$$

u is a viscosity subsolution iff for every smooth test-function ϕ touching u from above in some point x_0, we have

$$-G(x_0, \phi, D\phi, D^2\phi) \leq 0.$$

Finally, u is a viscosity solution if it is both a viscosity super- and subsolution. Back

Consider viscosity solutions to the problem

$$(\partial_t u - F) = 0.$$

Assume that u is a subsolution of the problem with initial condition $u(0, \cdot) = u_0$ and v is a supersolution with initial condition $v(0, \cdot) = v_0$. The problem satisfies comparison iff

$$u_0 \leq v_0 \Rightarrow u \leq v \text{ on } [0, T] \times \mathbb{R}^n.$$

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