Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Computational finance – Lecture 6

Christian Bayer

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 0 · www.wias-berlin.de
May 21, 2021

$$\mathrm{d}X_t = V(X_t)\mathrm{d}t + \sum_{i=1}^{d} V_i(X_t)\mathrm{d}W_t^i, \quad X_0 = x_0 \in \mathbb{R}^n$$

$$\overline{X}_0 := x_0, \quad \overline{X}_{t_{j+1}} := \overline{X}_{t_j} + V\left(\overline{X}_{t_j}\right)\Delta t_j + \sum_{i=1}^{d} V_i\left(\overline{X}_{t_i}\right)\Delta W_j^i, \quad j = 0, \dots, N-1$$

- Strong convergence with rate $1/2$: Suppose that $V, V_1, \dots, V_d$ are Lipschitz, then

$$E\left[\sup_{0\le t\le T}\left|X_t - \overline{X}_t\right|\right] \le C\sqrt{|\mathcal{D}|}.$$

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Theorem (Cameron and Clark, 1980)

Let $\mathcal{D}_N := \{0, T/N, \dots, T\}$ *and* $\mathcal{G}_{\mathcal{D}} := \sigma(\{W_t \mid t \in \mathcal{D}\})$. *Consider the system*

$$\mathrm{d}X_t^1 = \mathrm{d}W_t^1, \quad \mathrm{d}X_t^2 = X_t^1\mathrm{d}W_t^2, \quad X_0 = 0.$$

Then $E\left[\left|X_T^2 - E[X_T^2 \mid \mathcal{G}_{\mathcal{D}_N}]\right|^2\right]^{1/2} = \frac{T}{2}N^{-1/2}$.

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$$Lh(x) := \sum_{k=1}^{n} V^k(x)\partial_k h(x) + \frac{1}{2}\sum_{l,k=1}^{n} a^{kl}(x)\partial_{kl}h(x), \quad a^{kl}(x) := \sum_{i=1}^{d} V_i^k(x)V_i^l(x)$$

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Kolmogorov backward equation

$$\partial_t u(t,x) + Lu(t,x) = 0, \quad u(T,x) = f(x)$$

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Theorem (Weak convergence – version 1)

Assume that $V, V_1, \ldots, V_d$ are C^∞-bounded, and $\mathcal{G} = C^\infty_{\text{pol}}$. Then the Euler scheme converges weakly with rate 1, i.e.,

$$\forall f \in \mathcal{G} : \quad e(h, f) := \left| E\left[f\left(\overline{X}_T\right)\right] - E\left[f(X_T) \right] \right| \le Ch.$$

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Moreover, there is an error representation

$$e(h, f) = h \int_0^T E[\psi_1(s, X_s)]ds + h^2 e_2(T, f) + \mathcal{O}(h^3),$$

where $\psi_1(t, x) = \frac{1}{2} \sum_{i,j=1}^n V^i(x) V^j(x) \partial_{(i,j)} u(t, x) + \frac{1}{2} \sum_{i,j,k=1}^n V^i(x) a^j_k(x) \partial_{(i,j,k)} u(t, x) + \cdots$

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Theorem

$$\overline{R}(h) := \overline{A}(h/2) + \frac{\overline{A}(h/2) - \overline{A}(h)}{2^n - 1} = \frac{2^n \overline{A}(h/2) - \overline{A}(h)}{2^n - 1}$$

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Proof.

$$\overline{R}(h) = \frac{2^n[A - C(h/2)^n + \mathcal{O}(h^m)] - [A - Ch^n + \mathcal{O}(h^m)]}{2^n - 1} = A + \mathcal{O}(h^m).$$

□

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- Let $X_s^{t,x}$, $t \leq s \leq T$, denote the solution of the SDE started at $X_t^{t,x} = x$. By the Markov property, $u(t,x) = E[f(X_T) \mid X_t = x] = E[f(X_T^{t,x})]$.

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- $J_{t\to s}(x) := \frac{\partial}{\partial x} X_s^{t,x}$ by formally differentiating the SDE:

$$dJ_{t\to s}(x) = DV(X_s^{t,x})J_{t\to s}(x)ds + \sum_{i=1}^{d} DV_i(X_s^{t,x})J_{t\to s}(x)dW_s^i, \quad J_{t\to t}(x) = \mathrm{Id}_n$$

- Note that pair $(X_\cdot^{t,x}, J_{t\to\cdot}(x))$ solves SDE.

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- Note that pair $(X_\cdot^{t,x}, J_{t\to\cdot}(x))$ solves SDE.
- Now differentiate inside the expectation.

Lemma

$$E\left[u(t_{i+1},\overline{X}_{t_{i+1}}) \mid \overline{X}_{t_i} = x\right] = u(t_i,x) + h^2\psi_1(t,x) + O(h^3)$$

Proof – 3

- Proof only used first five (mixed) moments of (ΔW_j^i), $1 \le j \le N$, $1 \le i \le d$. Hence, weak schemes can be used, e.g. ΔW_j^i i.i.d. copies of $\sqrt{h}Y$,

$$Y = \begin{cases} \sqrt{3}, & \text{with probability } 1/6, \\ 0, & \text{with probability } 2/3, \\ -\sqrt{3}, & \text{with probability } 1/6. \end{cases}$$

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$X = X_T$ is the solution of an SDE, $f : \mathbb{R}^n \to \mathbb{R}$ a payoff. Compute $E\left[f(X_T)\right] = I[f; X_T]$.

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Euler – Monte Carlo method

$$E\left[f(X_T)\right] \approx I_M\left[f; \overline{X}_T\right], \quad \overline{X}_T \text{ based on grid of size } N.$$

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- Computing $E[f(\overline{X}_T)]$ is a $N \times d$-dimensional integration problem (difficult for QMC?).
- Error decomposition:

$$\left|E[f(X_T)] - I_M\left[f; \overline{X}_T\right]\right| \le \underbrace{\left|E[f(X_T)] - E\left[f\left(\overline{X}_T\right)\right]\right|}_{=: e_{\text{disc}}} + \underbrace{\left|E\left[f\left(\overline{X}_T\right)\right] - I_M\left[f; \overline{X}_T\right]\right|}_{=: e_{\text{stat}}}$$

- Generically, $e_{\text{disc}} \lesssim C_{\text{disc}}/N$, $e_{\text{stat}} \lesssim C_{\text{stat}}/\sqrt{M}$. Hence, given error tolerance $\text{TOL} > 0$, choose $N \simeq \text{TOL}^{-1}$, $M \simeq \text{TOL}^{-2}$, leading to computational cost $\simeq \text{TOL}^{-3}$.