From Hamiltonian particle systems to Kinetic equations

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The Landau equations: grazing collision limit

Landau in 1936 introduced a new kinetic equation for a dense, weakly interacting gas. Actually he did it for a Coulomb plasma (with various cutoff).
Consider the collision operator in the form

$$Q(f,f) = \int dv_1 \int dp \quad w(p)\delta(p^2 + (v - v_1)\cdot p)[f' f_1' - f f_1]$$

where

$$f' = f(v+p), \qquad f_1' = f(v_1 - p)$$

being p the transferred momentum. w is spherically symmetric and smooth. δ assures the energy conservation. $\varepsilon > 0$ is a small parameter.

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being p the transferred momentum. w is spherically symmetric and smooth. δ assures the energy conservation. $\varepsilon > 0$ is a small parameter. The transferred momentum is small, we rescale w as $\frac{1}{\varepsilon^3} w(\frac{p}{\varepsilon})$. Concentrates on the grazing collisions. We also rescale the mean-free path inverse by a factor $\frac{1}{\varepsilon}$ to take into account the high density situation. The result is

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$$Q_\varepsilon(f,f) = \frac{1}{\varepsilon^4}\int dv_1 \int dp \quad w(\frac{p}{\varepsilon})\delta(p^2 + (v - v_1)\cdot p)[f' f_1' - f f_1] =$$

$$\frac{1}{2\pi\varepsilon^2}\int dv_1 \int dp \quad w(p)\int_{-\infty}^{+\infty} ds e^{is(p^2\varepsilon + (v-v_1)\cdot p)}$$

$$[f(v+\varepsilon p)f(v_1 - \varepsilon p) - f(v)f(v_1)] =$$

$$\frac{1}{2\pi\varepsilon}\int dv_1 \int dp \quad w(p)\int_0^1 d\lambda \int_{-\infty}^{+\infty} ds e^{is(p^2\varepsilon + (v-v_1)\cdot p)}$$

$$\frac{d}{d\lambda} f(v+\varepsilon\lambda p)f(v_1 - \varepsilon\lambda p) =$$

$$\frac{1}{2\pi\varepsilon}\int dv_1 \int dp \quad w(p)\int_0^1 d\lambda \int_{-\infty}^{+\infty} ds e^{is(p^2\varepsilon + (v-v_1)\cdot p)}$$

$$p\cdot(\nabla_v - \nabla_{v_1})f(v+\varepsilon\lambda p)f(v_1 - \varepsilon\lambda p).$$

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Let φ be a test function, then

$$(\varphi, Q_\varepsilon(f,f)) = \frac{1}{2\pi\varepsilon}\int dv \int dv_1 \int dp \quad w(p)\int_0^1 d\lambda \int_{-\infty}^{+\infty} ds$$
$$e^{is(p^2(\varepsilon-2\varepsilon\lambda)+(v-v_1)\cdot p)}\varphi(v-\varepsilon\lambda p)\, p\cdot(\nabla_v-\nabla_{v_1})ff_1 =$$
$$\frac{1}{2\pi\varepsilon}\int dv \int dv_1 \int dp \quad w(p)\int_0^1 d\lambda \int_{-\infty}^{+\infty} ds e^{is(v-v_1)\cdot p}$$
$$[\varphi(v)-\varepsilon p\cdot\nabla_v\varphi(v)] \quad p\cdot(\nabla_v-\nabla_{v_1})ff_1 +$$
$$\frac{1}{2\pi}\int dv \int dv_1 \int dp \quad w(p)\int_{-\infty}^{+\infty} ds e^{is(v-v_1)\cdot p}\varphi(v)$$
$$isp^2\int_0^1 d\lambda(1-2\lambda) \quad p\cdot(\nabla_v-\nabla_{v_1})ff_1 + O(\varepsilon)$$

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$$p \cdot \nabla_v \varphi \; p \cdot (\nabla_v - \nabla_{v_1}) f f_1 + O(\varepsilon).$$

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Therefore:

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$$Q_L(f,f) = \int dv_1 \nabla_v \; a(\nabla_v - \nabla_{v_1}) f f_1,$$

$a = a(v - v_1)$ is the matrix

$$a_{i,j}(V) = \int dp\; w(p)\; \delta(V \cdot p)\; p_i p_j .$$

The Landau equations: grazing collision limit

Also

$$a_{i,j}(V) = \frac{1}{|V|} \int dp \; |p| w(p) \; \delta(\hat{V} \cdot \hat{p}) \; \hat{p}_i \hat{p}_j$$
$$= \frac{B}{|V|} \int d\hat{p} \quad \delta(\hat{V} \cdot \hat{p}) \quad \hat{p}_i \hat{p}_j,$$

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$$a_{i,j}(V) = \frac{B}{|V|}(\delta_{i,j} - \hat{V}_i\hat{V}_j), \quad a(V) = \frac{B}{|V|}P_V^{\perp}.$$

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Same properties as for the Boltzmann equation.

$$(v^\alpha, Q_L(f,f)) = 0$$

for $\alpha = 0, 1, 2$ and the Entropy production is given by the following expression

$$-(\log f, Q_L(f,f)) = \frac{1}{2}\int dv \int dv_1 \frac{1}{ff_1}\frac{1}{|v-v_1|}|P^{\perp}_{v-v_1}(\nabla_v - \nabla_{v_1})ff_1|^2.$$

The weak coupling limit

N identical particles of unitary mass. Positions and velocities: $q_1 \dots q_N$, $v_1 \dots v_N$.

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Here $F = -\nabla \phi$, ϕ the smooth, two-body, spherically symmetric interaction potential and τ the time.

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Here $F = -\nabla\phi$, ϕ the smooth, two-body, spherically symmetric interaction potential and τ the time. In this regime N is very large and the interaction strength quite moderate. $\varepsilon > 0$ a small parameter = the ratio between the macro and microscales.
$N = O(\varepsilon^{-3})$, the density is $O(1)$.
Rescale $x = \varepsilon q, \qquad t = \varepsilon\tau, \quad \phi \to \sqrt{\varepsilon}\phi.$

$$\frac{d}{dt} x_i = v_i \qquad \frac{d}{dt} v_i = \frac{1}{\sqrt{\varepsilon}} \sum_{\substack{j=1\dots N: \\ j \neq i}} F\left(\frac{x_i - x_j}{\varepsilon}\right).$$

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The total momentum variation for unit time is $O(\frac{1}{\sqrt{\varepsilon}})$.
But zero in the average.
The variance $= \frac{1}{\varepsilon} O(\sqrt{\varepsilon})^2 = O(1)$.

The weak coupling limit

$$X_N = x_1 \dots x_N \quad V_N = v_1 \dots v_N.$$

Liouville equation

$$(\partial_t + V_N \cdot \nabla_N) W^N(X_N, V_N) = \frac{1}{\sqrt{\varepsilon}} (T_N^\varepsilon W^N)(X_N, V_N)$$

where $V_N \cdot \nabla_N = \sum_{i=1}^N v_i \cdot \nabla_{x_i}$

$$(T_N^\varepsilon W^N)(X_N, V_N) = \sum_{0<k<\ell\leq N} (T_{k,\ell}^\varepsilon W^N)(X_N, V_N),$$

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The weak coupling limit

BBKGY hierarchy of equations for the marginals f_j^N (for $1 \leq j \leq N$):

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_k) f_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^N + \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon f_{j+1}^N.$$

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$$C_{k,j+1}^\varepsilon f_{j+1}(x_1 \dots x_j; v_1 \dots v_j) =$$
$$- \int dx_{j+1} \int dv_{j+1} F(\frac{x_k - x_\ell}{\varepsilon}) \cdot \nabla_{v_k} f_{j+1}(x_1, x_2, \dots, x_{j+1}; v_1, \dots, v_{j+1}).$$

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The initial value $\{f_j^0\}_{j=1}^N$ factorizes

$$f_j^0 = f_0^{\otimes j}, \text{for some} \quad f_0.$$

The weak coupling limit

Duhamel formula:

$$(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j),$$

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$$(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j),$$

$$f_j^N(t) = S(t)f_j^0 + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) C_{j+1}^{\varepsilon} f_{j+1}^N(t_1) dt_1 + \\ \frac{1}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) T_j^{\varepsilon} f_j^N(t_1) dt_1.$$

Assuming that the time evolved j-particle distributions $f_j^N(t)$ are smooth

$$C_{j+1}^{\varepsilon} f_{j+1}^N(X_j; V_j; t_1) = \\ -\varepsilon^3 \sum_{k=1}^{j} \int dr \int dv_{j+1} F(r) \cdot \nabla_{v_k} f_{j+1}(X_j, x_k - \varepsilon r; V_j, v_{j+1}, t_1) = O(\varepsilon^4)$$

because $\int drF(r) = 0$. Also the third term is vanishing.

The weak coupling limit

Hence $f_j^N(t)$ cannot be smooth !
We conjecture

$$f_j^N = g_j^N + \gamma_j^N$$

where g_j^N is the main part of f_j^N and is smooth, while γ_j^N is small, but strongly oscillating.

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) g_j^N = \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^{\varepsilon} g_{j+1}^N + \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^{\varepsilon} \gamma_{j+1}^N$$

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) \gamma_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^{\varepsilon} \gamma_j^N + \frac{1}{\sqrt{\varepsilon}} T_j^{\varepsilon} g_j^N,$$

Initial data

$$g_j^N(X_j, V_j) = f_j^0(X_j, V_j), \quad \gamma_j^N(X_j, V_j) = 0.$$

Note that $\gamma_1^N = 0$ since $T_1^{\varepsilon} = 0$.

The weak coupling limit

The remarkable fact of this decomposition is that γ can be eliminated. Let $(X_j(t), V_j(t)) = (\{x_1(t) \dots x_j(t), v_1(t) \dots v_j(t)\})$ be the solution of the j-particle flow (in macro variables)

$$\frac{d}{dt} x_i = v_i \qquad \frac{d}{dt} v_i = -\frac{1}{\sqrt{\varepsilon}} \sum_{\substack{k=1\dots j: \\ k \neq i}} \nabla \phi\left(\frac{x_i - x_k}{\varepsilon}\right).$$

Initial datum $(X_j, V_j) = (\{x_1 \dots x_j, v_1 \dots v_j\})$. $U_j(t)$ is the operator solving the Liouville equation

$$(\partial_t + V_j \cdot \nabla_j) h(X_j, V_j; t) = \frac{1}{\sqrt{\varepsilon}} \left(T_j^{\varepsilon} h\right)(X_N, V_N; t)$$

namely

$$h(X_j, V_j, t) = U_j h(X_j, V_j) = h(X_j(-t), V_j(-t)).$$

The weak coupling limit

Then

$$\gamma_j^N(t) = -\frac{1}{\sqrt{\varepsilon}} \int_0^t ds U(s) T_j^\varepsilon g_j^N(t-s).$$

$$\gamma_j^N(X_j, V_j, t) = -\frac{1}{\sqrt{\varepsilon}} \int_0^t ds \sum_{1 \le i < k \le j} \nabla \phi\left(\frac{x_i(-s) - x_k(-s)}{\varepsilon}\right) \cdot$$

$$(\nabla_{v_i} - \nabla_{v_k}) g_j^N(X_j(-s), V_j(-s); t-s).$$

Finally we arrive to a closed hierarchy for g^N:

$$(\partial_t + \sum_{k=1}^j v_k \cdot \nabla_{x_k}) g_j^N(X_j, V_j; t) = \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon g_{j+1}^N(X_j, V_j; t) +$$

$$\frac{N-j}{\varepsilon} \sum_{k=1}^j \sum_{i,r=1}^{j+1} \int_0^t ds \int dv_{j+1} \int dx_{j+1} \mathrm{div}_{v_k} F\left(\frac{x_k - x_{j+1}}{\varepsilon}\right) F\left(\frac{x_i(-s) - x_r(-s)}{\varepsilon}\right)$$

$$(\nabla_{v_i} - \nabla_{v_r}) g_{j+1}^N(X_{j+1}(-s), V_{j+1}(-s); t-s).$$

The weak coupling limit

We now present a formal derivation of the Landau eq.n (assuming g_2^Nsmooth).

$$(\partial_t + v_1 \cdot \nabla_{x_1}) g_1^N(t) = \frac{N-1}{\sqrt{\varepsilon}} C_2^\varepsilon g_2^N(t) + \frac{N-1}{\varepsilon} C_2^\varepsilon \int_0^t ds U_2(s) T_2 g_2^N(t-s).$$

Let $u \in \mathcal{D}$ be a test function.

$$\frac{N-1}{\varepsilon}(u, C_2^\varepsilon g_2^N(t)) = O(\sqrt{\varepsilon}).$$

The weak coupling limit

Last term:

$$-\frac{N-1}{\varepsilon}\int dx_1\int dx_2\int dv_1\int dv_2\int_0^t ds \quad \nabla_{v_1}u(x_1,v_1)$$

$$F(\frac{x_1-x_2}{\varepsilon})F(\frac{x_1(-s)-x_2(-s)}{\varepsilon})\cdot(\nabla_{v_1}-\nabla_{v_2})g_2^N(X_2(-s),V_2(-s);t-s)\approx$$

$$-\int dx_1\int dr\int dv_1\int dv_2\int_0^\infty ds \quad \nabla_{v_1}u(x_1,v_1)$$

$$F(r)F(\frac{x_1(-\varepsilon s)-x_2(-\varepsilon s)}{\varepsilon})\cdot(\nabla_{v_1}-\nabla_{v_2})g_2^N(x_1,x_2,v_1,v_2;t).$$

$(r=\frac{x_1-x_2}{\varepsilon})$ and $s\to\frac{s}{\varepsilon}$.

The weak coupling limit

$w = v_1 - v_2$ the relative velocity:

$$\frac{x_1(-\varepsilon s) - x_2(-\varepsilon s)}{\varepsilon} = r + ws + \frac{1}{\varepsilon}\int_0^{-\varepsilon s} d\tau (v_1(\tau) - v_1) - (v_2(\tau) - v_2).$$

But

$$v_1(\tau) - v_1 = \frac{1}{\sqrt{\varepsilon}}\int_0^{\tau} ds F(\frac{x_1(s) - x_2(s)}{\varepsilon}) = O(\sqrt{\varepsilon}).$$

The time spent when the two particles are at distance less that ε is $O(\varepsilon)$, (if the relative velocity w not too small). Thus:

$$\approx - \int dx_1 \int dr \int dv_1 \int dv_2 \int_0^{\infty} ds \quad \nabla_{v_1} u(x_1, v_1) F(r) F(r + ws)$$
$$(\nabla_{v_1} - \nabla_{v_2}) g_2^N(x_1, x_1, v_1, v_2; t)$$
$$\approx (u, Q_L(g_1^N, g_1^N)).$$

Invoking propagation of chaos.

The weak coupling limit

Actually it can be proven that

$$\int dr \int_0^\infty ds F(r) F(r - ws) = \frac{1}{2} \int dr \int_{-\infty}^\infty ds F(r) F(r - ws) = a(w)$$

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$$a(w)_{\alpha,\beta} = \frac{B}{|w|} (\delta_{\alpha,\beta} - \frac{w_\alpha w_\beta}{|w|^2})$$

and

$$B = C \int_0^\infty d\rho \rho^3 \hat{\phi}^2(\rho).$$

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The weak coupling limit

Consider the first order (in time) approximation $\tilde{g}^N$of g^N:

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k})\tilde{g}_j^N(X_j, V_j; t) = \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^{\varepsilon} S(t) f_{j+1}^0(X_j, V_j) +$$

$$\frac{N-j}{\varepsilon} \sum_{k=1}^{j} \sum_{i,r=1}^{j+1} \int_0^t ds \int dv_{j+1} \int dx_{j+1} \mathrm{div}_{v_k} F(\frac{x_k - x_{j+1}}{\varepsilon}) F(\frac{x_i(-s) - x_r(-s)}{\varepsilon})$$

$$(\nabla_{v_i} - \nabla_{v_r}) S(t-s) f_{j+1}^0(X_{j+1}(-s), V_{j+1}(-s)).$$

The weak coupling limit

Bobylev, P. and Saffirio 2012: derivation...... at time zero

Theorem

Suppose $f_0 \in C_0^3(\mathbb{R}^3 \times \mathbb{R}^3)$ *be the initial probability density satisfying:*

$$|D^r f_0(x,v)| \leq Ce^{-b|v|^2} \quad \textit{for} \qquad r = 0,1,2 \tag{1}$$

where D^r *is any derivative of order* r *and* $b > 0$. $\phi \in C^2(\mathbb{R}^3)$, $\phi \geq 0$ *and* $\phi(x) = 0$ *if* $|x| > 1$. *Assume factorization at time zero, then*

$$\lim_{\varepsilon \to 0} \tilde{g}_1^N(t) = S(t)f_0 + \int_0^t d\tau S(t-\tau)Q_L(S(\tau)f_0, S(\tau)f_0)$$

where $N\varepsilon^3 = 1$ *and the above limit is considered in* $\mathcal{D}'$.

The weak coupling limit

Propagation of chaos

Theorem

Under the same hypotheses

$$\lim_{\varepsilon\to 0} \tilde{g}_j^N(t, x_1, v_1, \dots, x_j, v_j) = \prod_{i=1}^{j} S(t) f_0(x_i, v_i)$$

$$+\sum_{i=1}^{j} \prod_{\substack{k=1 \\ k\neq i}}^{j} S(t) f_0(x_k, v_k) \int_0^t d\tau S(t-\tau) Q_L(S(\tau) f_0, S(\tau) f_0)(x_i, v_i)$$

in $\mathcal{D}'$.