Switched Systems Optimal Control Problems

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Overview

(1) Introduction

(2) Problem formulation

(3) SSOCP with fixed sequence of phases

(4) SSOCP with unknown sequence of phases

Introduction

Switched System

A dynamics system that operates by switching between different subsystems or phases.

Switched System Optimal Control Problem

Problem of designing an optimal sequence of phases and optimal control signals for each phase, such that certain cost function is minimized.

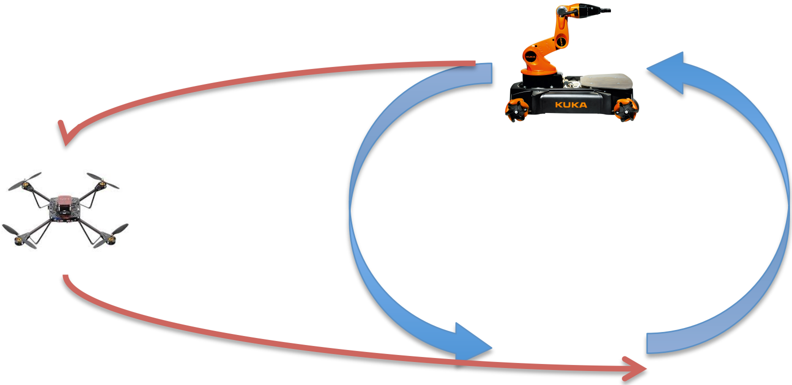
Applications:

- Aircraft modelling
- Air traffic control
- Robotics and industrial processes
- Logistics

Introduction

Example Problem

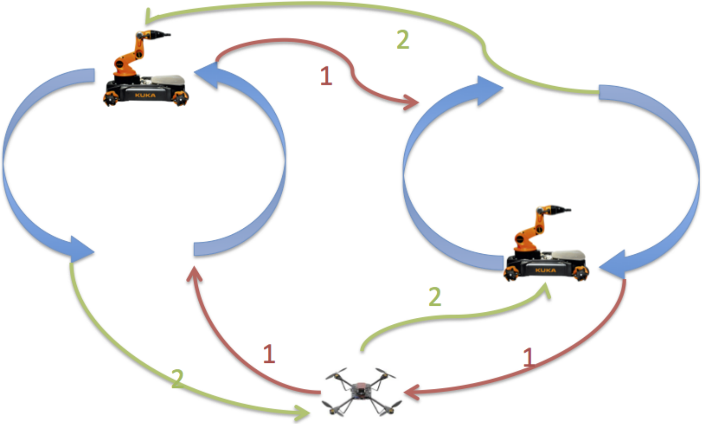
Interaction between quadcopter and robot.



Introduction

Example

Quadcopter flying over several industrial robots



Problem Formulation

Given M phase intervals $[T_0, T_1], [T_1, T_2], \ldots, [T_{M-1}, T_M]$ with length $p_1, p_2, \ldots, p_M$, we consider the following Switched System Optimal Control Problem

Switched System Optimal Control Problem

For each phase $[T_{k-1}, T_k]$

$$\textit{minimize} \quad \varphi^k(x(T_{k-1}), x(T_k)) \tag{1}$$

with respect to $x \in W^{1,\infty}([T_{k-1}, T_k]; \mathbb{R}^{n_x})$, $u \in L^\infty([T_{k-1}, T_k]; \mathbb{R}^{n_u^k})$ and $p_k \in \mathbb{R}$, subject to

$$\dot{x}(t) - f^k(t, x(t), u(t)) = 0_{\mathbb{R}^{n_x}} \quad a.a.\ t \in (T_{k-1}, T_k) \tag{2}$$

$$g^k(x(t), u(t)) \leq 0_{\mathbb{R}^{n_g^k}} \quad \forall\ t \in (T_{k-1}, T_k) \tag{3}$$

$$\phi^k(x(T_{k-1}), x(T_k)) = 0_{\mathbb{R}^{n_\phi^k}} \tag{4}$$

Problem Formulation

Two cases may occur:

1) The sequence of phases is fixed:
    - SSOCP can be transformed to standard optimal control problem
    - Solvable by gradient type methods (SQP, Interior point, Quasi-Newton)
    - Easier to implement and solve
2) The sequence of phases is unknown:
    - Optimal Control Problem for each phase
    - Feasible and infeasible sequences of phases may arise
    - Two phases may not occur simultaneously
    - Need of binary (decision) variables
    - Mixed-Integer solvers have to be used

SSOCP with fixed sequence of phases

Since the length of the time intervals $[T_{k-1}, T_k]$ is unknown, for each $k \in \{1, \ldots, M\}$ we consider the time transformation

$$t^{(k)} : [0, 1] \to [T_{k-1}, T_k]$$

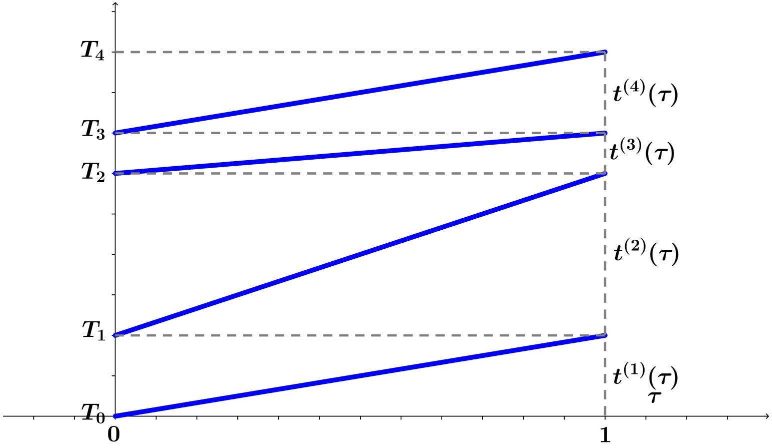
defined as

$$t^{(k)}(\tau) = T_{k-1} + \tau \cdot (T_k - T_{k-1}) = T_{k-1} + \tau \cdot p_k. \tag{5}$$

Note that the transformation $t^{(k)}(\cdot)$ is continuous and it holds

$$\frac{dt^{(k)}(\tau)}{d\tau} = T_k - T_{k-1} = p_k \quad \forall \tau \in (0, 1) \tag{6}$$

SSOCP with fixed sequence of phases



SSOCP with fixed sequence of phases

Let us define the new dimensions

$$N_x = M \cdot n_x \quad \text{and} \quad N_u = \sum_{k=1}^{M} n_u^k \tag{7}$$

and the new states and controls

$$x \in W^{1,\infty}([0,1];\mathbb{R}^{N_x}) \qquad x = \left(x^{(1)},\ldots,x^{(M)}\right) \qquad x^{(k)} \in W^{1,\infty}([0,1];\mathbb{R}^{n_x}) \tag{8}$$

$$u \in L^{\infty}([0,1];\mathbb{R}^{N_u}) \qquad u = \left(u^{(1)},\ldots,u^{(M)}\right) \qquad u^{(k)} \in L^{\infty}([0,1];\mathbb{R}^{n_u^k}) \tag{9}$$

SSOCP with fixed sequence of phases

In the same way, we redefine the functions involved in the Switched System Optimal Control Problem as follows.

Cost Function Reformulation

Define

$$\varphi : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}$$

such that

$$\varphi(x, y) = \sum_{k=1}^{M} \varphi^k(x^{(k)}, y^{(k)}) \tag{10}$$

for every $x = (x^{(1)}, \ldots, x^{(M)})$ and $y = (y^{(1)}, \ldots, y^{(M)})$ in $\mathbb{R}^{N_x}$.

SSOCP with fixed sequence of phases

Dynamics Reformulation

Define

$$f : (0,1) \times \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \longrightarrow \mathbb{R}^{N_x}, \quad f = \left(f^{(1)}, \ldots, f^{(M)}\right) \tag{11}$$

where for every $k = 1, \ldots, M$, the k^{th} component

$$f^{(k)} : (0,1) \times \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \longrightarrow \mathbb{R}^{n_x} \tag{12}$$

is defined as

$$f^{(k)}(\tau, x, u) = f^k(t^{(k)}(\tau), x^{(k)}, u^{(k)}) \tag{13}$$

for every $\tau \in (0,1)$, $x = \left(x^{(1)}, \ldots, x^{(M)}\right) \in \mathbb{R}^{N_x}$ and $u = \left(u^{(1)}, \ldots, u^{(M)}\right) \in \mathbb{R}^{N_u}$.

SSOCP with fixed sequence of phases

Constraints Reformulation

Define $N_g = \sum_{k=1}^{M} n_g^k$ and consider the function

$$g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{N_g}, \quad g = (g^{(1)}, \ldots, g^{(M)}) \tag{14}$$

where for every $k = 1, \ldots, M$, the k^{th} component

$$g^{(k)} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x^g}$$

is defined as

$$g^{(k)}(x, u) = g^k(x^{(k)}, u^{(k)}) \tag{15}$$

for every $x = (x^{(1)}, \ldots, x^{(M)}) \in \mathbb{R}^{n_x}$ and $u = (u^{(1)}, \ldots, u^{(M)}) \in \mathbb{R}^{n_u}$.

SSOCP with fixed sequence of phases

Boundary Conditions Reformulation

Define $N_\phi = \sum_{k=1}^{M} n_\phi^k$ and consider the function

$$\phi : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_\phi}, \quad \phi = \left(\phi^{(1)}, \dots, \phi^{(M)}\right) \tag{16}$$

where for every $k = 1, \dots, M$, the k^{th} component

$$\phi^{(k)} : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}^{n_x^\phi}$$

is defined as

$$\phi^{(k)}(x, y) = \phi^k(x^{(k)}, y^{(k)}) \tag{17}$$

for every $x = \left(x^{(1)}, \dots, x^{(M)}\right)$ and $y = \left(y^{(1)}, \dots, y^{(M)}\right)$ in $\mathbb{R}^{N_x}$.

SSOCP with fixed sequence of phases

Remark

Note that additional boundary conditions have to be imposed, in case continuity of the states between the different phases is not ensured by ϕ^k. Indeed, in that case we consider the function

$$\Phi : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}^{(M-1)\cdot n_x} \tag{18}$$

defined as

$$\Phi(x, y) = \begin{bmatrix} x^{(2)} - y^{(1)} \\ \vdots \\ x^{(M)} - y^{(M-1)} \end{bmatrix} \tag{19}$$

for every $x = \left(x^{(1)}, \dots, x^{(M)}\right)$ and $y = \left(y^{(1)}, \dots, y^{(M)}\right)$ in $\mathbb{R}^{N_x}$.

SSOCP with fixed sequence of phases

Finally, let us define the $N_x \times N_x$ diagonal matrix

$$P = \begin{bmatrix} p_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & p_1 & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & p_M & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & p_M \end{bmatrix} \tag{20}$$

SSOCP with fixed sequence of phases

With the previous reformulation, the Switched System Optimal Control Problem becomes

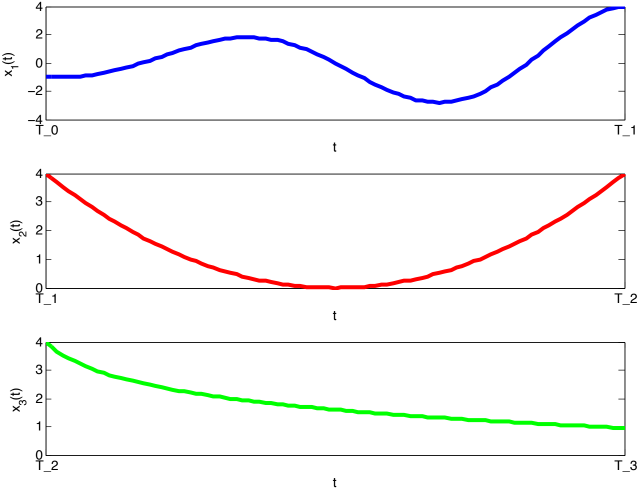
Reformulated Optimal Control Problem

$$\textit{Minimize} \quad \varphi(x(0), x(1))$$

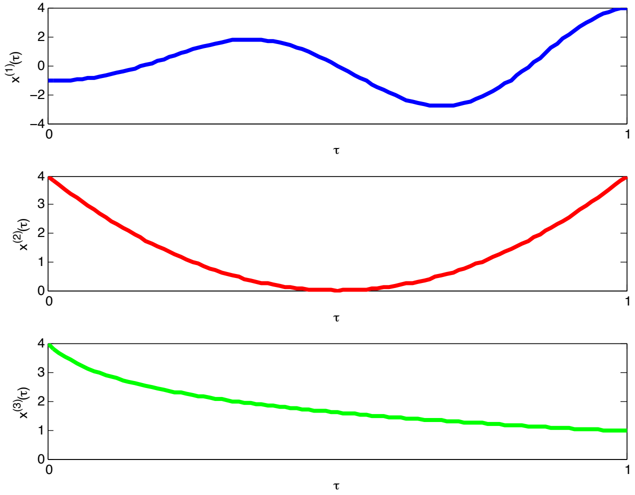
with respect to $x \in W^{1,\infty}([0,1]; \mathbb{R}^{N_x})$, $u \in L^{\infty}([0,1]; \mathbb{R}^{N_u})$ and $p = (p_1, \ldots, p_M) \in \mathbb{R}^M$, subject to

$$\begin{aligned}
\dot{x}(\tau) - P \cdot f(\tau, x(\tau), u(\tau)) &= 0_{\mathbb{R}^{N_x}} && \text{a.a. } \tau \in (0,1) \\
g(x(\tau), u(\tau)) &\leq 0_{\mathbb{R}^{N_g}} && \forall\, \tau \in (0,1) \\
\phi(x(0), x(M)) &= 0_{\mathbb{R}^{N_\phi}} \\
\Phi(x(0), x(M)) &= 0_{\mathbb{R}^{(M-1)\cdot n_x}}
\end{aligned}$$

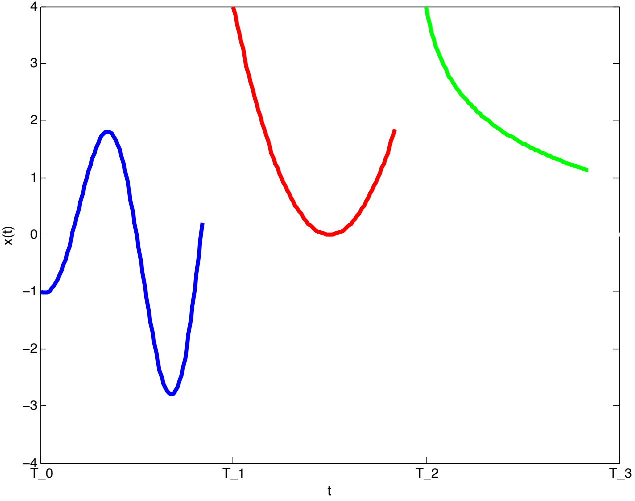
SSOCP with fixed sequence of phases



SSOCP with fixed sequence of phases



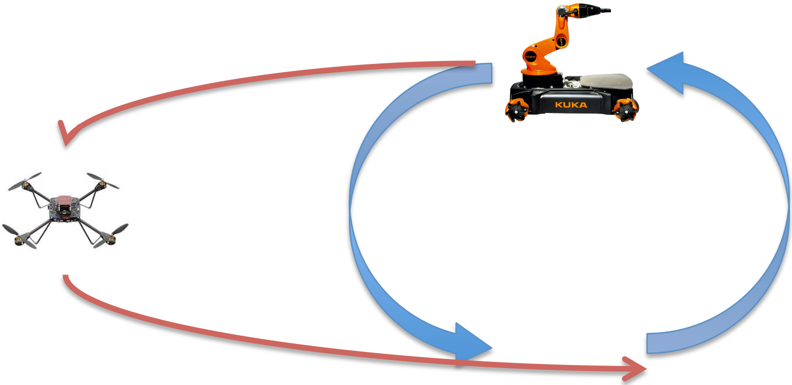
SSOCP with fixed sequence of phases



SSOCP with fixed sequence of phases

Example Problem

Interaction between quadcopter and youBot platform.



Robot Model

We consider a model of the omni-directional mobile platform youBot provided by KUKA, consisting of:

1) youBot omni-directional platform
   - 4 KUKA omniWheels
   - 3 degrees of freedom
2) youBot arm with gripper
   - 5 axes robot arm
   - 5 degrees of freedom
   - 2 finger gripper

Robot Base

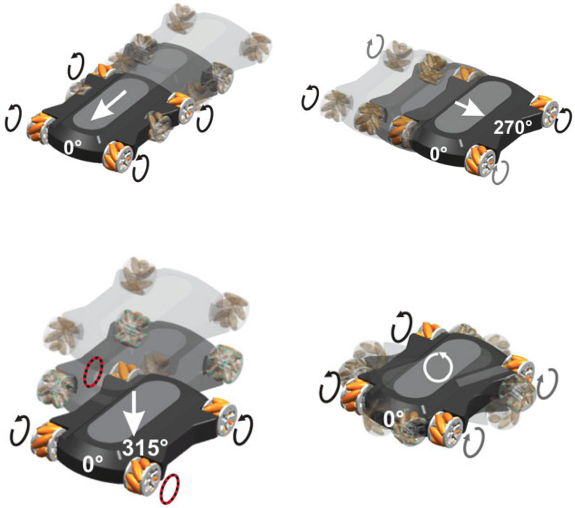
Base Model

- 3 states: x, y and φ representing the position of the robot base in the $xy-$plane and its orientation
- 3 states: v_x, v_y and v_φ representing the velocities of the base (translational and angular)
- 3 controls: u_x, u_y and u_φ representing the accelerations (translational and angular)

Equations of Motion

$$\begin{aligned}
\dot{x}(t) &= v_x(t) \\
\dot{y}(t) &= v_y(t) \\
\dot{\varphi}(t) &= v_\varphi(t) \\
\dot{v}_x(t) &= u_x(t)\cos(\varphi(t)) + u_y(t)\sin(\varphi(t)) \\
\dot{v}_y(t) &= u_x(t)\sin(\varphi(t)) - u_y(t)\cos(\varphi(t)) \\
\dot{v}_\varphi(t) &= u_\varphi(t)
\end{aligned} \tag{21}$$

Robot Base



Robot Arm

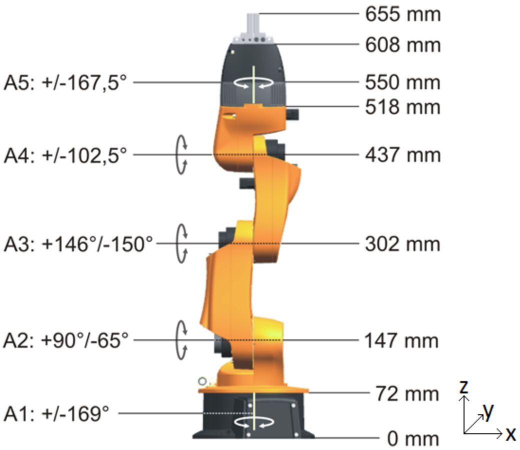
Arm Model

- 5 states: $q_1, \ldots, q_5$ representing the angles of the joints
- 5 states: $v_1, \ldots, v_5$ representing the angular velocities of the joints
- 5 controls: $u_1, \ldots, u_5$ representing the angular acceleration of the joints

Equations of Motion

$$\begin{aligned} \dot{q}_1(t) &= v_1(t), & \dot{v}_1(t) &= u_1(t) \\ \dot{q}_2(t) &= v_2(t), & \dot{v}_2(t) &= u_2(t) \\ \dot{q}_3(t) &= v_3(t), & \dot{v}_3(t) &= u_3(t) \\ \dot{q}_4(t) &= v_4(t), & \dot{v}_4(t) &= u_4(t) \\ \dot{q}_5(t) &= v_5(t), & \dot{v}_5(t) &= u_5(t) \end{aligned} \tag{22}$$

Robot Arm


655 mm
608 mm
A5: +/-167,5°
550 mm
518 mm
A4: +/-102,5°
437 mm
A3: +146°/-150°
302 mm
A2: +90°/-65°
147 mm
72 mm
A1: +/-169°
0 mm
z
y
x

Robot Arm

Let r be the offset vector of the first joint with respect to the base and let $l_1, \ldots, l_4$ be the lengths of the four arms. We define the rotation matrices

$$S_0(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad S_1(\beta) = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}. \tag{23}$$

$$\begin{aligned} S_{01}(\alpha, \beta) &= S_0(\alpha)S_1(\beta) \\ S_{012}(\alpha, \beta, \gamma) &= S_0(\alpha)S_1(\beta)S_1(\gamma) \\ S_{0123}(\alpha, \beta, \gamma, \delta) &= S_0(\alpha)S_1(\beta)S_1(\gamma)S_1(\delta) \end{aligned} \tag{24}$$

Then, the mount points $P_1(q), \ldots, P_4(q)$ and the gripper position $P_5(q)$ are given by the following equations

Robot Arm

Gripper Position

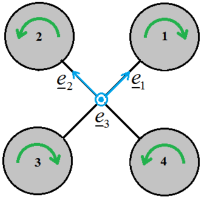
$$
\begin{aligned}
P_1(q) &= S_0(q_1)r \\
P_2(q) &= P_1(q) + S_{01}(q_1, q_2)\begin{pmatrix} 0 \\ 0 \\ l_1 \end{pmatrix} \\
P_3(q) &= P_2(q) + S_{012}(q_1, q_2, q_3)\begin{pmatrix} 0 \\ 0 \\ l_2 \end{pmatrix} \\
P_4(q) &= P_3(q) + S_{0123}(q_1, q_2, q_3, q_4)\begin{pmatrix} 0 \\ 0 \\ l_3 \end{pmatrix} \\
P_5(q) &= P_4(q) + S_{0123}(q_1, q_2, q_3, q_4)\begin{pmatrix} 0 \\ 0 \\ l_4 \end{pmatrix}
\end{aligned}
\tag{25}
$$

Quadcopter Model

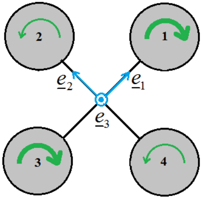
Ultralight UAV with four rotors:

1) Six states:
   - x, y, z position
     on the quadcopter
   - yaw, roll and pitch angle
2) Four controls:
   - RPM of the rotors

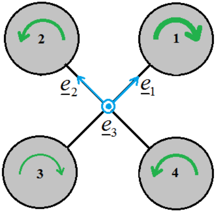
Quadcopter Model



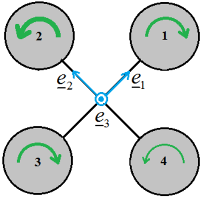
(a) Neutral position



(b) Pos. rotation around e_3



(c) Neg. rotation around e_2



(d) Pos. rotation around e_1

Quadcopter Model

Equations of Motion

$$m \cdot \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = F_A + F_m + F_W$$

where

F_A :	lift force generated by the rotors
F_m :	gravitational force
F_W :	Drag

Quadcopter Model

Lift generated by the rotors

$$F_A = A\left(N_{Blades} \cdot C_A \cdot \frac{1}{2} \cdot \rho(z) \cdot A_{Blades} \cdot \begin{bmatrix} 0 \\ 0 \\ U_1^2 + U_2^2 + U_3^2 + U_4^2 \end{bmatrix}\right)$$

Drag force

$$F_W = C_W \cdot \frac{1}{2} \cdot \rho(z) \begin{bmatrix} -sign(v_x) \cdot v_x^2 \cdot A_{eff,x} \\ -sign(v_y) \cdot v_y^2 \cdot A_{eff,y} \\ -sign(v_z) \cdot v_z^2 \cdot A_{eff,z} \end{bmatrix}$$

Quadcopter Model

Moment generated by the Lift force

$$M_A = A\left(\frac{1}{4}\cdot\rho(z)\cdot N_{Blades}\cdot C_A\cdot r^2\cdot d\cdot\begin{bmatrix} U_2^2 - U_4^2 \\ U_3^2 - U_1^2 \\ 0 \end{bmatrix}\right)$$

Moment generated by the rotors

$$M_R = A\left(\rho(z)\cdot N_{Blades}\cdot A_{Blades}\cdot C_M\cdot r^3\cdot\begin{bmatrix} 0 \\ 0 \\ U_1^2 - U_2^2 + U_3^2 - U_4^2 \end{bmatrix}\right)$$

Applications

Problem 1: Two interacting robots

- approach phase
- interaction phase
- return phase
- solution first case
- solution second case

Problem 2: Robot intercepted by a quadcopter

- approach phase
- fly-over phase
- return phase
- solution

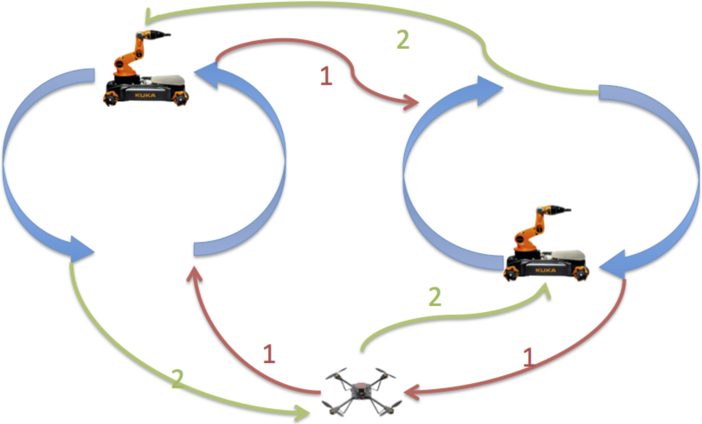
SSOCP with unknown sequence of phases

- Each phase is an Optimal Control Problem
- Sequencing the single phases
- Not every sequence of phases is feasible

SSOCP with unknown sequence of phases

Example

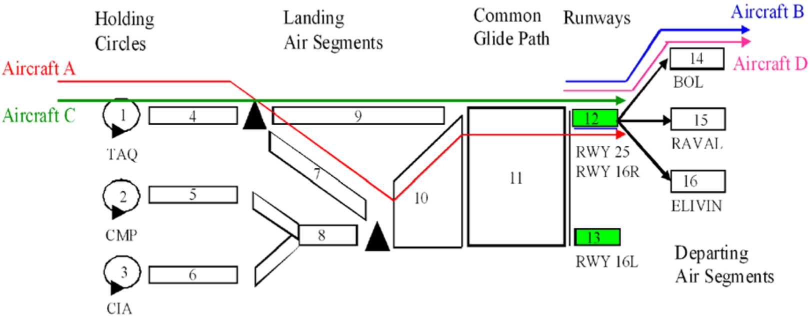
Quadcopter flying over several industrial robots



SSOCP with unknown sequence of phases

Example

Air traffic scheduling and control



SSOCP with unknown sequence of phases

(1) BILEVEL OPTIMIZATION APPROACH

- Fixed sequence of phases are transformed to standard Optimal Control Problem (OCP)
- Mixed-Integer Optimization Problem (MIOP) is formulated for the unknown sequence of phases
- Bilevel Optimization Problem is solved with MIOP as upper level problem and OCP as lower level one

(2) EQUILIBRIUM CONSTRAINTS APPROACH

- Fixed sequence of phases are transformed to standard OCP and first order necessary conditions are formulated
- MIOP is formulated for the unknown sequence of phases
- Large scale Mixed-Integer Mathematical Program with Equilibrium Constraints is solved

Decision Variables

Problem:

- Two phases Φ_1, Φ_2 with starting times T_1, T_2 and phase lengths p_1, p_2
- One of the states only starts when the other one is finished

Approach:

- Define "big" constant $N = T_1 + T_2 + p_1 + p_2$
- Define decision variable $x \in \{0, 1\}$
- Consider the constraints:

$$T_1 + p_1 - T_2 \leq (1 - x) \cdot N$$
$$T_2 + p_2 - T_1 \leq x \cdot N$$

corresponding to

$$x = 1 \quad \Leftrightarrow \quad \Phi_1 \text{ occures before } \Phi_2$$
$$x = 0 \quad \Leftrightarrow \quad \Phi_2 \text{ occures before } \Phi_1$$

Bilevel Optimization Approach

Denote with $OCP(k)$ the Optimal Control Problem corresponding to the k-th phase ($k = 1, 2$), i.e.

$OCP(k)$

$$\textit{Minimize} \quad \varphi^k(x(T_k), x(T_k + p_k)) \tag{26}$$

with respect to $x \in W^{1,\infty}([T_k, T_k + p_k]; \mathbb{R}^{n_x})$, $u \in L^\infty([T_k, T_k + p_k]; \mathbb{R}^{n_u^k})$ and $p_k \in \mathbb{R}$, subject to

$$\dot{x}(t) - f^k(t, x(t), u(t)) = 0_{\mathbb{R}^{n_x}} \quad a.a.\ t \in (T_k, T_k + p_k) \tag{27}$$

$$g^k(x(t), u(t)) \leq 0_{\mathbb{R}^{n_g^k}} \quad \forall\ t \in (T_k, T_k + p_k) \tag{28}$$

$$\phi^k(x(T_k), x(T_k + p_k)) = 0_{\mathbb{R}^{n_\phi^k}} \tag{29}$$

Bilevel Optimization Approach

Let $p_1^*(T_1)$ and $p_2^*(T_2)$ be the optimal phase length of $OCP(1)$ and $OCP(2)$ respectively, and let $\frac{\partial p_1^*}{\partial T_1}(T_1)$ and $\frac{\partial p_2^*}{\partial T_2}(T_2)$ be their sensitivities.

MIOP

$$\text{Minimize} \quad \frac{\partial p_1^*}{\partial T_1}(T_1) \cdot d_1 + \frac{\partial p_2^*}{\partial T_2}(T_2) \cdot d_2 \tag{30}$$

with respect to $d_1, d_2 \in \mathbb{R}$ and $x \in \{0, 1\}$, subject to

$$T_1 + p_1^*(T_1) + \frac{\partial p_1^*}{\partial T_1}(T_1) \cdot d_1 - T_2 \leq (1 - x) \cdot N \tag{31}$$

$$T_2 + p_2^*(T_2) + \frac{\partial p_2^*}{\partial T_2}(T_2) \cdot d_2 - T_1 \leq x \cdot N \tag{32}$$

Bilevel Optimization Approach

Algorithm

(1) Choose feasible starting times T_1 and T_2 and phase lengths p_1 and p_2

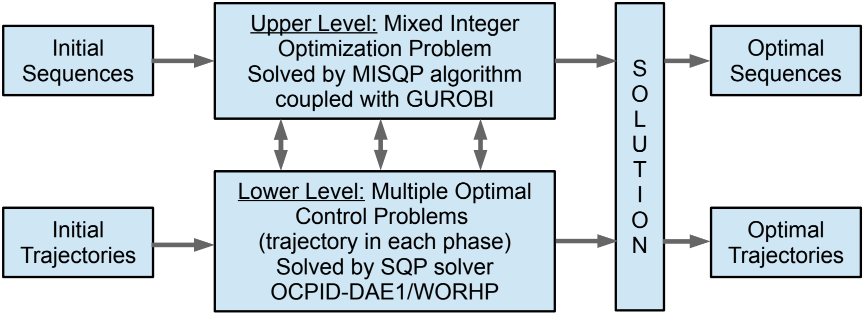
(2) Solve $OCP(k)$ for each phase k, compute sensitivities

(3) Evaluate stopping criteria

(4) Solve $MIOP$, compute search directions d_1 and d_2

(5) Update $T_1 = T_1 + d_1$, $T_2 = T_2 + d_2$ and GOTO (2)

Bilevel Optimization Approach



Equilibrium Constraints Approach

Discretization

Define the discretization grid

$$\mathbb{G}_k = \left\{ t_0 + i \cdot h_k \;\middle|\; t_0 = T_k, h_k = \frac{p_k}{N_k}, i = 0, \ldots, N_k \right\}$$

where $N_k \in \mathbb{N}$. Let $x_i = x(t_i)$, $u_i = u(t_i)$ for $i = 0, \ldots, N_k$ be the discretized states and controls on $\mathbb{G}_k$.

$DOCP(k)$

$$\textit{Minimize} \quad \varphi^k(x_0, x_{N_k}) \tag{33}$$

with respect to $x_i \in \mathbb{R}^{n_x}$, $u_i \in \mathbb{R}^{n_u^k}$ for $i = 0, \ldots, N_k$ and $p_k \in \mathbb{R}$, subject to

$$x_{i+1} - x_i - f^k(t_i, x_i, u_i) = 0_{\mathbb{R}^{n_x}} \quad \forall\, i = 0, \ldots, N_k - 1 \tag{34}$$

$$g^k(x_i, u_i) \leq 0_{\mathbb{R}^{n_g^k}} \quad \forall\, i = 1, \ldots, N_k - 1 \tag{35}$$

$$\phi^k(x_0, x_{N_k}) = 0_{\mathbb{R}^{n_\phi^k}} \tag{36}$$

Equilibrium Constraints Approach

Lagrangian Function

We define the Lagrangian function of the *DOCP*(k)

$$\mathcal{L}_k : \mathbb{R}^{(N_k+1)\cdot n_x} \times \mathbb{R}^{N_k \cdot n_u^k} \times \mathbb{R}^{N_k \cdot n_x} \times \mathbb{R}^{(N_k-1)\cdot n_g} \times \mathbb{R}^{n_\phi} \to \mathbb{R}$$

as

$$\mathcal{L}_k(x, u, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma}) = \varphi^k(x_0, x_{N_k}) + \sum_{i=0}^{N_k-1} \boldsymbol{\lambda}_{i+1}^T \cdot \left[x_{i+1} - x_i - f^k(t_i, x_i, u_i)\right]$$
$$+ \sum_{i=1}^{N_k-1} \boldsymbol{\mu}_i^T \cdot g^k(x_i, u_i) + \boldsymbol{\sigma}^T \cdot \phi^k(x_0, x_{N_k}) \tag{37}$$

Equilibrium Constraints Approach

First Order Necessary Conditions for *DOCP*(k)

Let (x^*, u^*, p_k^*) be an optimal solution of *DOCP*(k), than there exist multipliers $\boldsymbol{\lambda}^* \in \mathbb{R}^{N_k \cdot n_x}$, $\boldsymbol{\mu}^* \in \mathbb{R}^{(N_k-1)\cdot n_g}$ and $\boldsymbol{\sigma}^* \in \mathbb{R}^{n_\phi}$, such that

$$\nabla_x \mathcal{L}_k(x^*, u^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\sigma}^*) = 0_{\mathbb{R}^{n_x}} \tag{38}$$

$$x_{i+1} - x_i - f^k(t_i, x_i, u_i) = 0_{\mathbb{R}^{n_x}} \quad \forall\, i = 0, \dots, N_k - 1 \tag{39}$$

$$g^k(x_i, u_i) \leq 0_{\mathbb{R}^{n_g^k}} \quad \forall\, i = 1, \dots, N_k - 1 \tag{40}$$

$$\boldsymbol{\mu}_i \geq 0_{\mathbb{R}^{n_g^k}} \quad \forall\, i = 1, \dots, N_k - 1 \tag{41}$$

$$\boldsymbol{\mu}_i^T \cdot g^k(x_i, u_i) = 0 \quad \forall\, i = 1, \dots, N_k - 1 \tag{42}$$

$$\phi^k(x_0, x_{N_k}) = 0_{\mathbb{R}^{n_\phi^k}} \tag{43}$$

Equilibrium Constraints Approach

Mixed-Integer Mathematical Program with Equilibrium Constraints

$$\textit{Minimize} \quad p_1 + p_2 \tag{44}$$

with respect to

$$T_1, T_2, p_1, p_2 \in \mathbb{R}$$

$$x \in \{0, 1\}$$

$$(x^{(k)}, u^{(k)}, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\mu}^{(k)}, \boldsymbol{\sigma}^{(k)}) \in \mathbb{R}^{(N_k+1)\cdot n_x} \times \mathbb{R}^{N_k \cdot n_u^k} \times \mathbb{R}^{N_k \cdot n_x} \times \mathbb{R}^{(N_k-1)\cdot n_g} \times \mathbb{R}^{n_\phi}$$

subject to

$$T_1 + p_1 - T_2 \leq (1 - x) \cdot N \tag{45}$$

$$T_2 + p_2 - T_1 \leq x \cdot N \tag{46}$$

$$\textit{First Order Necessary Conditions for each phase} \tag{47}$$

Summary

- Introduction and Problem formulation
- SSOCP with fixed sequence of phases
- Models and test problems
- SSOCP with unknown sequence of phases
- Bilevel optimization approach
- Equilibrium constraints approach

Thank you for your attention!

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