A common approach to singular perturbation and homogenization

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We consider periodic homogenization of boundary value problems for second-order semilinear elliptic systems in 2D of the type

$$\partial_{x_i} a_{ij}^{\alpha\beta}(x/\varepsilon)\partial_{x_j} u^\beta(x) = b^\alpha(x, u(x)) \text{ for } x \in \Omega.$$

For small $\varepsilon > 0$ we prove existence of weak solutions $u = u_\varepsilon$ as well as their local uniqueness for $\|u - u_0\|_\infty \approx 0$, where u_0 is a given non-degenerate weak solution to the homogenized boundary value problem, and we estimate the rate of convergence to zero of $\|u_\varepsilon - u_0\|_\infty$ for $\varepsilon \to 0$.

Our assumptions are, roughly speaking, as follows: The coefficients $a_{ij}^{\alpha\beta}$ are bounded, measurable and $\mathbb{Z}^2$-periodic, the maps $b^\alpha(\cdot, u)$ are bounded and measurable, the maps $b^\alpha(x, \cdot)$ are C^1-smooth, and Ω is a bounded Lipschitz domain in $\mathbb{R}^2$. Neither global solution uniqueness is supposed nor growth restriction of $b^\alpha(x, \cdot)$, and cross-diffusion is allowed.

The main tool of the proofs is an abstract result of implicit function theorem type which in the past has been applied to singularly perturbed nonlinear ODEs and elliptic and parabolic PDEs and, hence, which permits a common approach to existence, local uniqueness and error estimates for singularly perturbed problems and and for homogenization problems.

In order to apply implicit function theorems one needs isomorphism properties of the linearized operators. In our case they follow from K. Grögers result about maximal regularity of boundary value problems for elliptic systems with non-smooth data in the pair of Sobolev spaces $W_0^{1,p}(\Omega; \mathbb{R}^n)$ and $W^{-1,p}(\Omega; \mathbb{R}^n)$ (in the case of Dirichlet problems) with $p \approx 2$. In order to apply implicit function theorems one needs also C^1-smoothness of the appearing nonlinear superposition operators. In our case these operators have to be well-defined and C^1-smooth on $W^{1,p}(\Omega; \mathbb{R}^n)$ with $p > 2$, but $p \approx 2$, and therefore we have to suppose that the space dimension is two.

This is common work with N.N. Nefedov, cf. arXiv:2309.14108.