Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Extinction versus explosion in a supercritical super-Wright-Fisher diffusion

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submitted: 04 July 2002

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No. 752
Berlin 2002

1991 *Mathematics Subject Classification.* 60J80 60G57 35K55 35K15 60J57 60J60.

Key words and phrases. Binary splitting, binary branching, weighted superprocess, semilinear Cauchy problem, semilinear parabolic PDE, finite ancestry property, trimmed tree, h-transform, extinction, explosion .

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Abstract

We study mild solutions u to the semilinear Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}u_t(x) = \frac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2}u_t(x) + \gamma u_t(x)(1-u_t(x)) \qquad (t \geq 0), \\ u_0(x) = f(x), \end{cases}$$

with $x \in [0,1]$, f a nonnegative measurable function and γ a positive constant. Solutions to this equation are given by $u_t = \mathcal{U}_t f$, where $(\mathcal{U}_t)_{t\geq 0}$ is the log-Laplace semigroup of a supercritical superprocess taking values in the finite measures on $[0,1]$, whose underlying motion is the Wright-Fisher diffusion. We establish a dichotomy in the long-time behavior of this superprocess. For $\gamma \leq 1$, the mass in the interior $(0,1)$ dies out after a finite random time, while for $\gamma > 1$, the mass in $(0,1)$ explodes as time tends to infinity with positive probability. In the case of explosion, the mass in $(0,1)$ grows exponentially with rate $\gamma - 1$ and is approximately uniformly distributed over $(0,1)$. We apply these results to show that $(\mathcal{U}_t)_{t\geq 0}$ has precisely four fixed points when $\gamma \leq 1$ and five fixed points when $\gamma > 1$, and determine their domains of attraction.

Contents

1 Introduction and main results

1.1 Superprocesses and binary splitting particle systems

Before turning to our specific processes of interest, we recall the concept of a superprocess and a binary splitting particle system in a somewhat more general set-up. Let E be a compact metrizable space, let $B(E)$, $\mathcal{C}(E)$ denote the spaces of bounded measurable real functions and continuous real functions on E, respectively, and set $B_+(E) := \{f \in B(E) : f \geq 0\}$, etc. Let G be the generator of a Feller process $\xi = (\xi_t)_{t\geq 0}$ on E and let $\alpha \in \mathcal{C}_+(E)$, $\beta \in \mathcal{C}(E)$. Then, for each $f \in B_+(E)$, the semilinear Cauchy problem in $B_+(E)$

$$\begin{cases} \frac{\partial}{\partial t}u_t = Gu_t + \beta u_t - \alpha u_t^2 \qquad (t \geq 0), \\ u_0 = f, \end{cases} \tag{1}$$

has a unique mild solution $u_t =: \mathcal{U}_t f$ (see Section 2.1 for details). Moreover, there exists a unique (in law) Markov process $\mathcal{X}$ with continuous sample paths in the space $\mathcal{M}(E)$ of finite measures on E, defined by its Laplace functionals

$$E^{\mu}[e^{-\langle \mathcal{X}_t, f\rangle}] = e^{-\langle \mu, \mathcal{U}_t f\rangle} \qquad (t \geq 0,\ \mu \in \mathcal{M}(E),\ f \in B_+(E)). \tag{2}$$

$\mathcal{X}$ is called the *superprocess* in E with *underlying motion generator* G, *activity* α and *growth parameter* β (the last two terms are our terminology), or shortly the (G, α, β)*-superprocess.* $(\mathcal{U}_t)_{t\geq 0} = \mathcal{U} = \mathcal{U}(G, \alpha, \beta)$ is called the *log-Laplace semigroup* of $\mathcal{X}$. $\mathcal{X}$ can be constructed in several ways and is nowadays standard; see, e.g., [Fit88, Fit91, Fit92]. We can think of $\mathcal{X}$ as describing a population where mass flows with generator G, and during a time interval $\mathrm{d}t$ a bit of mass $\mathrm{d}m$ at position x produces offspring with mean $(1 + \beta(x)\mathrm{d}t)\mathrm{d}m$ and finite variance $2\alpha(x)\mathrm{d}t\,\mathrm{d}m$. For basic facts on superprocesses we refer to [Daw93, Eth00].

Similarly, when G is (again) the generator of a Feller process on a compact metrizable space E and $\gamma \in \mathcal{C}_+(E)$, then, for any $f \in B_{[0,1]}(E) := \{f \in B(E) : 0 \leq f \leq 1\}$, the semilinear Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u_t = G u_t + \gamma u_t(1 - u_t) \qquad (t \geq 0), \\ u_0 = f, \end{cases} \tag{3}$$

has a unique mild solution $u_t =: U_t f$ in $B_{[0,1]}(E)$. Moreover, there exists a unique Markov process X with cadlag sample paths in the space $\mathcal{N}(E)$ of finite counting measures on E, defined by its generating functionals

$$E^{\nu}\big[(1 - f)^{X_t}\big] = (1 - U_t f)^{\nu} \qquad (t \geq 0,\ \nu \in \mathcal{N}(E),\ f \in B_{[0,1]}(E)). \tag{4}$$

Here if $\nu = \sum_{i=1}^{n} \delta_{x_i}$ is a finite counting measure and $g \in B_{[0,1]}(E)$, then $g^{\nu} := \prod_{i=1}^{n} g(x_i)$. We call X the binary splitting particle system in E with underlying motion generator G and splitting rate γ, or shortly the (G, γ)*-bin-split-process.* $(U_t)_{t\geq 0} = U = U(G, \gamma)$ is called the *generating semigroup* of X. We interpret a counting measure $\sum_{i=1}^{n} \delta_{x_i}$ as a collection of particles, situated at positions $x_1, \ldots, x_n$. In this interpretation, the particles in X perform independent motions with generator G and additionally, a particle splits with local rate γ into two new particles, created at the position of the old one.

1.2 Introduction of the problem and motivation

Let $\overline{A}$ be the closure of the operator

$$A = \tfrac{1}{2} x(1 - x) \tfrac{\partial^2}{\partial x^2}. \tag{5}$$

Then $\overline{A}$ is the generator of a Feller process ξ on $[0, 1]$, called the (standard) *Wright-Fisher diffusion* (see [EK86, Theorem 8.2.8]). We are interested in mild solutions to the Cauchy equation

$$\begin{cases} \frac{\partial}{\partial t} u_t = \overline{A} u_t + \gamma u_t(1 - u_t) \qquad (t \geq 0), \\ u_0 = f, \end{cases} \tag{6}$$

where $\gamma > 0$ is a constant. For $f \in B_+[0, 1]$, the mild solution of (6) is given by $u_t = \mathcal{U}_t f$, where $\mathcal{U} = \mathcal{U}(\overline{A}, \gamma, \gamma)$ is the log-Laplace semigroup of a superprocess $\mathcal{X}$ in $[0, 1]$ with underlying motion generator $G = \overline{A}$, and activity and growth parameter $\alpha = \beta = \gamma$. We call $\mathcal{X}$ the *super-Wright-Fisher diffusion* (with activity and growth parameter $\gamma > 0$).[1] If $f \in B_{[0,1]}[0, 1]$, then

[1]More generally, if $\mathcal{Y}$ is the $(\overline{A}, \alpha, \gamma)$-superprocess, with $\alpha, \gamma > 0$ constants, then $\frac{\gamma}{\alpha}\mathcal{Y} = \mathcal{X}$ in law, and therefore this more general case can be reduced to the case $\alpha = \gamma$.

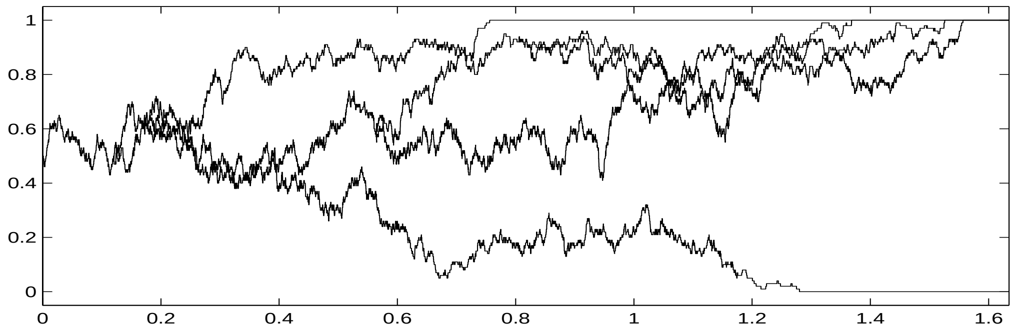

Figure 1: A system of binary splitting Wright-Fisher diffusions with splitting rate $\gamma = 1$.

the solution of (6) is also given by $u_t = U_t f$, where $U = U(\overline{A}, \gamma)$ is the generating semigroup of a system X of *binary splitting Wright-Fisher diffusions*, with splitting rate γ. See Figure 1 for a simulation of X for $\gamma = 1$. The points $0, 1$ are accessible traps for the Wright-Fisher diffusion, and therefore a natural question is whether eventually all particles of X end up in 0 or 1. This question will be answered for all $\gamma > 0$ in Proposition 4 below.

Our interest in the Cauchy equation (6) is motivated by recent work of Greven, Klenke and Wakolbinger [GKW01]. They study a system of linearly interacting Wright-Fisher diffusions on $\mathbb{Z}^d$, catalyzed by a voter model. They show that the long-time behavior of this model in dimension $d = 2$ can be expressed in terms of a function p, which is defined in terms of the system X of binary splitting Wright-Fisher diffusions with splitting rate $\gamma = 1$, as

$$p(x) := \lim_{t\to\infty} P^{\delta_x}[X_t(\{1\}) > 0] = \lim_{t\to\infty} P^{\delta_x}[X_t((0,1]) > 0] \qquad (x \in [0,1]). \tag{7}$$

In order to show that the two expressions for p in (7) are identical, they note that both expressions correspond to a fixed point p of the generating semigroup $U(\overline{A}, 1)$ with boundary conditions $p(0) = 0$ and $p(1) = 1$. Assuming that p is sufficiently smooth, the fixed point property means that p solves the equation

$$\tfrac{1}{2}x(1-x)\tfrac{\partial^2}{\partial x^2}p(x) + \gamma p(x)(1-p(x)) = 0 \qquad (x \in [0,1]). \tag{8}$$

Though stated only for the case $\gamma = 1$, the proof of Lemma 1.13 in [GKW01] shows that equation (8) has at most one solution with boundary conditions $p(0) = 0$ and $p(1) = 1$ when $\gamma < z_0^2/8 \cong 1.836$, where z_0 is the smallest non-trivial zero of the Bessel function of the first kind with parameter 1. The authors do not answer the question whether solutions to (8) with these boundary condions are unique for $\gamma \geq z_0^2/8$, or what solutions may exist for other boundary conditions. Proposition 3 below settles these questions. We show moreover that all fixed points of $U(\overline{A}, \gamma)$ are smooth, a fact tacitly assumed in [GKW01].

In the present paper, we prefer to study solutions to the Cauchy problem (6) by means of the $(\overline{A}, \gamma, \gamma)$-superprocess $\mathcal{X}$ rather than by means of the $(\overline{A}, \gamma)$-bin-split process X. The fact that $\mathcal{X}$ and X are related to the same semilinear Cauchy equation reflects a deeper fact, namely, that X is the *trimmed tree* of the superprocess $\mathcal{X}$. Heuristically, this means that the particles in X correspond to those infinitesimal bits of mass in $\mathcal{X}$, that have offspring at all later times. For a precise statement of this fact we refer the reader to our forthcoming paper [FS02a].

Another motivation for our present paper comes from work in progress on the renormalization analysis of systems of linearly interacting Wright-Fisher diffusions, catalyzed by an autonomous system of linearly interacting Wright-Fisher diffusions [FS02b]. In this project, there have turned up certain continuous-mass branching processes in discrete time, as well as their log-Laplace semigroups, which are in some sense similar to $\mathcal{U}(\overline{A},\gamma,\gamma)$, but more complicated to describe. In future, we hope to apply the methods developed in the present paper to study fixed points of these more complicated log-Laplace semigroups.

1.3 Results

The following theorem is our main result. We write 'eventually' behind an event, depending on t, to denote the existence of a (random) time $\tau<\infty$ such that the event holds for all $t\geq\tau$.

Theorem 1 (Long-time behavior of the super-Wright-Fisher diffusion) *Let $\mathcal{X}$ be the super-Wright-Fisher diffusion with activity and growth parameter equal to the same constant $\gamma>0$, started in $\mu\in\mathcal{M}[0,1]$. Set*

$$v(x):=6x(1-x)\qquad(x\in[0,1]).\tag{9}$$

Then there exist nonnegative random variables W_0, W_1, $W_{(0,1)}$ (depending on μ) such that

$$\begin{array}{lll}\text{(i)}\ \lim_{t\to\infty}e^{-\gamma t}\langle\mathcal{X}_t,1_{\{r\}}\rangle=W_r & \text{a.s.} & (r=0,1),\\ \text{(ii)}\ \lim_{t\to\infty}e^{-(\gamma-1)t}\langle\mathcal{X}_t,v\rangle=W_{(0,1)} & \text{a.s.} & \end{array}\tag{10}$$

and

$$\begin{array}{lll}\text{(i)}\ \{W_r=0\}=\{\mathcal{X}_t(\{r\})=0\ \textit{eventually}\} & \text{a.s.} & (r=0,1),\\ \text{(ii)}\ \{W_{(0,1)}=0\}=\{\mathcal{X}_t((0,1))=0\ \textit{eventually}\} & \text{a.s.} & \end{array}\tag{11}$$

Moreover,

$$\{W_{(0,1)}>0\}\subset\{W_1>0\}\cap\{W_2>0\}\quad\text{a.s.}\tag{12}$$

If $\gamma\leq 1$, then

$$W_{(0,1)}=0\qquad\text{a.s.}\tag{13}$$

If $\gamma>1$, then $W_{(0,1)}$ satisfies

$$E^{\mu}(W_{(0,1)})=\langle\mu,v\rangle\quad\textit{and}\quad\mathrm{Var}^{\mu}(W_{(0,1)})\leq 3\tfrac{\gamma}{\gamma-1}\langle\mu,v\rangle\tag{14}$$

as well as

$$\lim_{t\to\infty}E^{\mu}\Big[\big|e^{-(\gamma-1)t}\langle\mathcal{X}_t,vf\rangle-W_{(0,1)}\langle\ell,vf\rangle\big|^2\Big]=0\quad\forall f\in B[0,1],\tag{15}$$

where ℓ denotes the Lebesgue measure on $(0,1)$.

Theorem 1 has the following consequence for the log-Laplace semigroup $\mathcal{U}(\overline{A},\gamma,\gamma)$.

Proposition 2 (Long-time behavior of $\mathcal{U}(\overline{A},\gamma,\gamma)$) *Let $\mathcal{X}$, $W_0, W_1, W_{(0,1)}$ be as in Theorem 1 and let $\mathcal{U} = \mathcal{U}(\overline{A},\gamma,\gamma)$. Then, for all $\mu \in \mathcal{M}[0,1]$ and $f \in B_+[0,1]$,*

$$
\begin{aligned}
&\lim_{t\to\infty} e^{-\langle \mu, \mathcal{U}_t f\rangle} \\
&= P^{\mu}\Big[\big\{f(0)=0 \text{ or } W_0=0\big\}\cap\big\{f(1)=0 \text{ or } W_1=0\big\}\cap\big\{\langle \ell,f\rangle=0 \text{ or } W_{(0,1)}=0\big\}\Big] \\
&= \begin{cases}
1 & \text{if } f(0)=f(1)=\langle \ell,f\rangle=0,\\
P^{\mu}\big[W_{(0,1)}=0\big] & \text{if } f(0)=f(1)=0,\ \langle \ell,f\rangle>0,\\
P^{\mu}\big[W_0=0\big]=P^{\mu}\big[W_0=W_{(0,1)}=0\big] & \text{if } f(0)>0,\ f(1)=0,\\
P^{\mu}\big[W_1=0\big]=P^{\mu}\big[W_1=W_{(0,1)}=0\big] & \text{if } f(0)=0,\ f(1)>0,\\
P^{\mu}\big[W_0=W_1=0\big]=P^{\mu}\big[W_0=W_1=W_{(0,1)}=0\big] & \text{if } f(0)>0,\ f(1)>0.
\end{cases}
\end{aligned}
\tag{16}
$$

Here $P^{\mu}[W_{(0,1)}=0]<1$ if and only if $\gamma>1$ and $\langle \mu, v\rangle > 0$.

Except for the statement about smoothness (of the functions $p_1,\ldots,p_5$ below), the following result is immediate from Theorem 1 and Proposition 2.

Proposition 3 (Fixed points of $\mathcal{U}(\overline{A},\gamma,\gamma)$) *For any $\gamma>0$, all fixed points of the log-Laplace semigroup $\mathcal{U}(\overline{A},\gamma,\gamma)$ are given by*

$$
\left.\begin{aligned}
p_1(x) &:= 0,\\
p_2(x) &:= -\log P^{\delta_x}[W_{(0,1)}=0],\\
p_3(x) &:= -\log P^{\delta_x}[W_0=0],\\
p_4(x) &:= -\log P^{\delta_x}[W_1=0],\\
p_5(x) &:= -\log P^{\delta_x}[W_0=W_1=0]
\end{aligned}\right\}\quad (x\in[0,1]).
\tag{17}
$$

Here $p_2=0$ if $\gamma\leq 1$, and $p_2>0$ on $(0,1)$ if $\gamma>1$. The functions $p_1,\ldots,p_5$ are twice continuously differentiable on $[0,1]$ and solve (8).

Since conversely, every nonnegative twice continuously differentiable solution to (8) is a fixed point of $\mathcal{U}(\overline{A},\gamma,\gamma)$, we see that (8) has precisely four solutions when $\gamma\leq 1$ and precisely five solutions when $\gamma>1$. Proposition 2 shows that $\mathcal{U}_t f$ converges pointwise as $t\to\infty$ to one of the functions $p_1,\ldots,p_5$, where the limit depends on the values of $f(0)$, $f(1)$ and $\langle \ell, f\rangle$. The functions $p_1,\ldots,p_5$ are $[0,1]$-valued and therefore fixed points of the generating semigroup $U(\overline{A},\gamma)$ as well. Our final result describes $p_1,\ldots,p_5$ in terms of the system X of binary splitting Wright-Fisher diffusions with splitting rate γ.

Proposition 4 (Fixed points of $U(\overline{A},\gamma)$) *The functions $p_1,\ldots,p_5$ in (17) satisfy*

$$
\left.\begin{aligned}
p_1(x) &= 0,\\
p_2(x) &= P^{\delta_x}[X_t((0,1))>0 \text{ eventually}],\\
p_3(x) &= P^{\delta_x}[X_t(\{0\})>0 \text{ eventually}] = P^{\delta_x}[X_t([0,1))>0 \text{ eventually}],\\
p_4(x) &= P^{\delta_x}[X_t(\{1\})>0 \text{ eventually}] = P^{\delta_x}[X_t((0,1])>0 \text{ eventually}],\\
p_5(x) &= 1
\end{aligned}\right\}\ (x\in[0,1]).
\tag{18}
$$

See Figure 2 for a plot of the functions p_2 and p_4 (for $\gamma=2$).

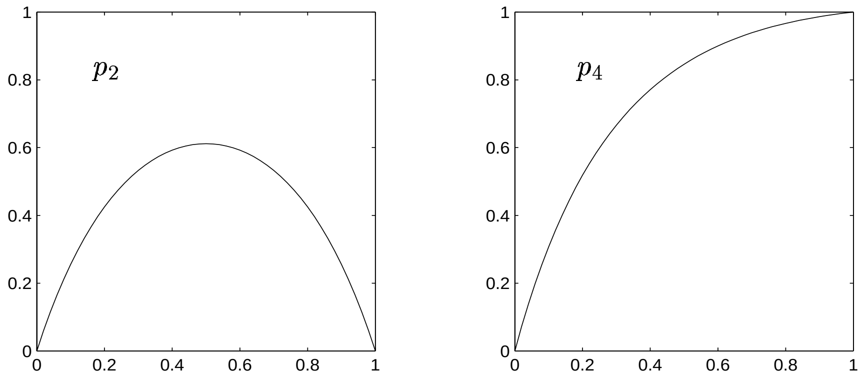

Figure 2: Two solutions to the differential equation $\frac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2}p(x) + 2p(x)(1-p(x)) = 0$.

1.4 Methods and related work

An essential tool in the proof of Theorem 1 is the *weighted super-Wright-Fisher diffusion* $\mathcal{X}^v$, defined as

$$\mathcal{X}^v_t(\mathrm{d}x) := v(x)\mathcal{X}_t(\mathrm{d}x) \qquad (t \geq 0), \tag{19}$$

where v is defined in (9). Note that v is an eigenfunction of the operator $\overline{A}$, with eigenvalue -1. For convenience, we have normalized v such that $\langle \ell, v \rangle = 1$.

In general, when a superprocess is weighted with a sufficiently smooth density, the result is a new superprocess, with a new activity and growth parameter and a new underlying motion, which is an h-transform of the old one. For the case that the underlying motion is a locally uniformly elliptic diffusion on $\mathbb{R}^d$, weighted superprocesses were developed by Engländer and Pinsky in [EP99]. In our case, the following can be proved without too much effort.

Lemma 5 (Weighted super-Wright-Fisher diffusion) *Let $\mathcal{X}$ be the super-Wright-Fisher diffusion with $\gamma > 0$ and let $\mathcal{X}^v$ be defined as in (19). Then $\mathcal{X}^v$ is the $(\overline{A^v}, \gamma v, \gamma - 1)$-superprocess in $[0,1]$, where $\overline{A^v}$ is the closure of the operator*

$$A^v := \tfrac{1}{2}x(1-x)\tfrac{\partial^2}{\partial x^2} + 2(\tfrac{1}{2} - x)\tfrac{\partial}{\partial x}. \tag{20}$$

Indeed, $\overline{A^v}$ generates a Feller process ξ^v in $[0,1]$ (see [EK86, Theorem 8.2.1]). The diffusion ξ^v is an h-transform (with $h = v$) of the Wright-Fisher diffusion ξ. This v-transformed Wright-Fisher diffusion ξ^v is ergodic with invariant law $v\ell$ (Lemma 20 below). For $\gamma > 1$, the $(\overline{A^v}, \gamma v, \gamma - 1)$-superprocess is supercritical, and in this case one expects $e^{-(\gamma-1)t}\mathcal{X}^v_t$ to converge, in some way, to a random multiple of $v\ell$. This is the idea behind formula (15).

Recently, Engländer and Turaev [ET02], have shown for a certain class of superdiffusions $\mathcal{X}$ in $\mathbb{R}^d$ with underlying motion generator G, growth parameter β and activity α, the convergence in law

$$e^{-\lambda_c t}\langle \mathcal{X}, g \rangle \Rightarrow W\langle \rho, g \rangle \quad \text{as } t \to \infty, \tag{21}$$

where W is a nonnegative random variable, λ_c is the generalized principal eigenvalue of $G + \beta$ (which is assumed to be positive), ρ is a measure on $\mathbb{R}^d$, defined in terms of $G + \beta$, and g is any compactly supported continuous function on $\mathbb{R}^d$. In their work, the weighted superprocess $\mathcal{X}^\phi_t(\mathrm{d}x) := \phi(x)\mathcal{X}_t(\mathrm{d}x)$ plays a central role, where ϕ is the principal eigenfunction of the operator $G+\beta$. Their dynamical system methods are based on a result on the existence of an invariant curve of the log-Laplace semigroup of their superprocess. Using this invariant curve, they give an expression for the Laplace-transform of the law of the random variable W in (21). Their methods use in an essential way the fact that their underlying space is $\mathbb{R}^d$ (and not an open subset of $\mathbb{R}^d$, like $(0,1)$), and therefore their results are not immediately applicable to our situation. It is stated as an open problem in [ET02] whether the random variable W in (21) in general satisfies $\{W=0\}=\{\mathcal{X}_t=0 \text{ eventually}\}$.

In our set-up, we can prove that $\{W_{(0,1)}=0\}=\{\mathcal{X}_t((0,1))=0 \text{ eventually}\}$ because of the following property of the weighted super-Wright-Fisher diffusion $\mathcal{X}^v$.

Lemma 6 (Finite ancestry) *For all $\gamma>0$, the weighted super-Wright-Fisher diffusion $\mathcal{X}^v$ satisfies*

$$\inf_{x\in[0,1]} P^{\delta_x}[\mathcal{X}^v_t=0]>0 \qquad \forall t>0. \tag{22}$$

Formula (22) has been called the *finite ancestry property* (of $\mathcal{X}^v$); for a justification of this terminology we refer the reader to [FS02a]. Our proof of Lemma 6 is quite long. It is not clear whether the weighted superprocesses $\mathcal{X}^\phi$ occurring in [ET02] will in general satisfy a formula of the form (22). Therefore, we mention as an open problem:

> How to check, in a practical way, whether a given superprocess has the finite ancestry property (22)?

Another problem that is left open by the present paper, is whether the L_2-convergence in (15) can be replaced by almost sure convergence. In fact, we suspect that (15) can be strengthened to

$$\lim_{t\to\infty} e^{-(\gamma-1)t}\langle\mathcal{X}_t,1_{(0,1)}f\rangle=W_{(0,1)}\langle\ell,f\rangle \quad \forall f\in B[0,1] \quad \text{a.s.}, \tag{23}$$

but we do not have a proof.

The rest of the paper is organized as follows. Sections 2.1 and 2.2 contain some general facts about (G,α,β)-superprocesses and (G,α,β)-superprocesses enjoying the finite ancestry property, respectively. After some preparatory work in Sections 2.3 and 2.4, we prove Lemmas 5 and 6 in Section 2.5. In Sections 2.6 and 2.7 we derive some properties of the weighted super-Wright-Fisher diffusion $\mathcal{X}^v$, culminating in the proof of Theorem 1 in Section 2.8. Finally, Sections 2.9–2.11 contain the proofs of Propositions 2, 4, and 3 (in this order).

2 Proofs

2.1 Preparation: some general facts about log-Laplace semigroups

Let E be a compact metrizable space and let $\mathcal{C}(E)$ be the space of continuous real functions on E, equipped with the supremum norm $\|\cdot\|_\infty$. Let $\xi=(\xi_t)_{t\geq0}$ be a Feller process in E with semigroup $S_tf(x):=E^x[f(\xi_t)]$ $(t\geq0,\ x\in E, f\in B(E))$. By definition, the (full) generator G of ξ is the linear operator on $\mathcal{C}(E)$ given by $Gf:=\lim_{t\to0}t^{-1}(S_tf-f)$ where the domain $\mathcal{D}(G)$ of G is the space of all functions $f\in\mathcal{C}(E)$ for which the limit exists in $\mathcal{C}(E)$.

Let $\alpha\in\mathcal{C}_+(E)$, $\beta\in\mathcal{C}(E)$, and $f\in\mathcal{C}_+(E)$. By definition, u is a *classical solution* of the Cauchy problem (1) if $u:[0,\infty)\to\mathcal{C}_+(E)\cap\mathcal{D}(G)$ is continuously differentiable in $\mathcal{C}(E)$ (i.e., the derivative $\frac{\partial}{\partial t}u_t:=\lim_{s\to t}s^{-1}(u_{t+s}-u_t)$ exists in $\mathcal{C}(E)$ for all $t\geq0$ and the map $\frac{\partial}{\partial t}u$:

$[0,\infty) \to \mathcal{C}(E)$ is continuous) and (1) holds. A measurable function $u : [0,\infty) \times E \to [0,\infty)$ is called a *mild solution* of (1) if u is bounded on finite time intervals and solves (pointwise)

$$u_t = S_t f + \int_0^t S_{t-s}\big(\beta u_s - \alpha u_s^2\big)\mathrm{d}s \qquad (t \geq 0). \tag{24}$$

Equation (1) has a unique mild solution for all $f \in B_+(E)$ (see [Fit88]) and this solution is a classical solution if $f \in \mathcal{C}_+(E) \cap \mathcal{D}(G)$ (see [Paz83, Theorems 6.1.4 and 6.1.5]; the fact that f is nonnegative and $\alpha \geq 0$ implies that solutions cannot explode).

The (G, α, β)-superprocess $\mathcal{X}$ is defined as the unique strong Markov process with continuous sample paths in $\mathcal{M}(E)$, equipped with the topology of weak convergence, such that (2) holds for all $f \in B_+(E)$; see [Fit88, Fit91, Fit92].

Note the following elementary properties of the log-Laplace semigroup $\mathcal{U}(G, \alpha, \beta)$. Here, we write $\text{bp-lim}_{n\to\infty} f_n = f$ if f is the bounded pointwise limit of the sequence $(f_n)_{n\geq 0}$.

Lemma 7 (Continuity and monotonicity of log-Laplace semigroups) *For each $t \geq 0$, $\mathcal{U}_t : \mathcal{C}_+(E) \to \mathcal{C}_+(E)$ is continuous. Moreover, if* $\text{bp-lim}_{n\to\infty} f_n = f$ *for some sequence $f_n \in B_+(E)$, then* $\text{bp-lim}_{n\to\infty} \mathcal{U}_t f_n = \mathcal{U}_t f$. *Finally, $f \leq g$ implies $\mathcal{U}_t f \leq \mathcal{U}_t g$ $(f, g \in B_+(E))$.*

Proof The continuity of $\mathcal{U}_t : \mathcal{C}_+(E) \to \mathcal{C}_+(E)$ follows from [Paz83, Theorem 6.1.2] and the fact that solutions do not explode. Continuity of $\mathcal{U}_t$ with respect to bounded pointwise limits is obvious from (2), and the same formula also makes clear that $\mathcal{U}_t : B_+(E) \to B_+(E)$ is monotone. ∎

Recall that (1) has a classical solution for $f \in \mathcal{C}_+(E) \cap \mathcal{D}(G)$. Because of the following, for many purposes it suffices to work with classical solutions.

Lemma 8 (Closure and bp-closure) *For $t \geq 0$ fixed, $\{(f, \mathcal{U}_t f) : f \in \mathcal{C}_+(E)\}$ is the closure in $\mathcal{C}(E)$ of $\{(f, \mathcal{U}_t f) : f \in \mathcal{C}_+(E) \cap \mathcal{D}(G)\}$, and $\{(f, \mathcal{U}_t f) : f \in B_+(E)\}$ is the bp-closure of $\{(f, \mathcal{U}_t f) : f \in \mathcal{C}_+(E)\}$.*

Here, the bp-closure of a set B is the smallest set $\overline{B}$ such that $B \subset \overline{B}$ and $f \in \overline{B}$ whenever $\text{bp-lim}_{n\to\infty} f_n = f$ for some sequence $f_n \in \overline{B}$.

Proof of Lemma 8 It follows from the Hille-Yosida Theorem (see [EK86, Theorem 1.2.6]) that $\mathcal{D}(G)$ is dense in $\mathcal{C}(E)$. Since $\mathcal{D}(G)$ is a linear space and $1 \in \mathcal{D}(G)$, it is not hard to see that $\mathcal{C}_+(E) \cap \mathcal{D}(G)$ is dense in $\mathcal{C}_+(E)$. The fact that $\{(f, \mathcal{U}_t f) : f \in \mathcal{C}_+(E)\}$ is the closure in $\mathcal{C}(E)$ of $\{(f, \mathcal{U}_t f) : f \in \mathcal{C}_+(E) \cap \mathcal{D}(G)\}$ now follows from the continuity of $\mathcal{U}_t : \mathcal{C}_+(E) \to \mathcal{C}_+(E)$.

In [EK86, Proposition 3.4.2] it is proved that $\mathcal{C}(E)$ is bp-dense in $B(E)$; the argument can easily be adapted to show that $\mathcal{C}_+(E)$ is bp-dense in $B_+(E)$. Therefore Lemma 8 follows from the continuity of $\mathcal{U}_t$ with respect to bounded pointwise limits. ∎

$\mathcal{U}_t f$ may be defined unambiguously such that (2) holds also for functions f that are not bounded, or even infinite.

Lemma 9 (Extension of $\mathcal{U}$ to unbounded functions) *For each measurable $f : E \to [0, \infty]$ and $t \geq 0$ there exists a unique measurable $\mathcal{U}_t f : E \to [0, \infty]$ such that (2) holds for all $\mu \in \mathcal{M}(E)$, where we put $e^{-\infty} := 0$.*

Proof Define $\mathcal{U}_t f$ by $\mathcal{U}_t f(x) := -\log E^{\delta_x}[e^{-\langle \mathcal{X}_t, f\rangle}]$ where $\log 0 := -\infty$. To see that (2) holds again for all $\mu \in \mathcal{M}(E)$, choose $B_+(E) \ni f_n \uparrow f$, note that $\mathcal{U}_t f_n \uparrow \mathcal{U}_t f$, and take the limit in (2). ∎

We will often need the following comparison result.

Lemma 10 (Sub- and supersolutions) *Assume that $T > 0$ and that $\tilde{u} : [0,T] \to \mathcal{C}_+(E) \cap \mathcal{D}(G)$ is continuously differentiable in $\mathcal{C}(E)$ and solves*

$$\tfrac{\partial}{\partial t}\tilde{u}_t \leq G\tilde{u}_t + \beta\tilde{u}_t - \alpha\tilde{u}_t^2 \qquad (t \in [0,T]). \tag{25}$$

Then $\tilde{u}_T \leq \mathcal{U}_T\tilde{u}_0$. The same holds with both inequality signs reversed.

Proof Let $g : [0,T] \to \mathcal{C}_+(E)$ be defined by the formula

$$\tfrac{\partial}{\partial t}\tilde{u}_t = G\tilde{u}_t + \beta\tilde{u}_t - \alpha\tilde{u}_t^2 - g_t \qquad (t \in [0,T]). \tag{26}$$

Set $u_t := \mathcal{U}_t\tilde{u}_0$. Then $u : [0,T] \to \mathcal{C}_+(E)$ is the classical solution of

$$\begin{cases} \tfrac{\partial}{\partial t}u_t = Gu_t + \beta u_t - \alpha u_t^2 & (t \in [0,T]), \\ u_0 = \tilde{u}_0. \end{cases} \tag{27}$$

Put $\Delta_t := u_t - \tilde{u}_t$ $(t \in [0,T])$. Then Δ solves

$$\begin{cases} \tfrac{\partial}{\partial t}\Delta_t = G\Delta_t + \beta\Delta_t - \alpha\,(u_t + \tilde{u}_t)\Delta_t + g_t & (t \in [0,T]), \\ \Delta_0 = 0. \end{cases} \tag{28}$$

The generator G satisfies the positive maximum principle (see [EK86, Theorem 4.2.2]) and therefore (28) implies that $\Delta \geq 0$. For imagine that $\Delta_t(x) < 0$ somewhere on $[0,T] \times E$. Let R be a constant such that $\beta - \alpha\,(u_t + \tilde{u}_t) + R > 0$. Then $\tilde{\Delta}_t := e^{Rt}\Delta_t$ solves

$$\begin{cases} \tfrac{\partial}{\partial t}\tilde{\Delta}_t = G\tilde{\Delta}_t + \{\beta - \alpha\,(u_t + \tilde{u}_t) + R\}\tilde{\Delta}_t + g_t e^{Rt} & (t \in [0,T]), \\ \tilde{\Delta}_0 = 0. \end{cases} \tag{29}$$

If $\tilde{\Delta}_t(x) < 0$ for some $(t,x) \in [0,T] \times E$, then $\tilde{\Delta}$ must assume a negative minimum over $[0,T] \times E$ in some point (s,y), with $s > 0$ since $\tilde{\Delta}_0 = 0$. But in such a point one would have $\frac{\partial}{\partial s}\tilde{\Delta}_s(y) \leq 0$ while $G\tilde{\Delta}_s(y) + \{\beta(y) - \alpha(y)\,(u_s(y) + \tilde{u}_s(y)) + R\}\tilde{\Delta}_s(y) + g_s(y)e^{Rs} > 0$, in contradiction with (29).

The same argument applies when both inequality signs are reversed. ∎

Lemma 10 has the following application.

Lemma 11 (Bounds on log-Laplace semigroups) *Let $\mathcal{U} = \mathcal{U}(G,\alpha,\beta)$, $\overline{\mathcal{U}} = \mathcal{U}(G,\underline{\alpha},\overline{\beta})$, where $\alpha,\underline{\alpha} \in \mathcal{C}_+(E)$ and $\beta,\overline{\beta} \in \mathcal{C}(E)$ satisfy*

$$\alpha \geq \underline{\alpha} \quad \textit{and} \quad \beta \leq \overline{\beta}. \tag{30}$$

Then

$$\mathcal{U}_t f \leq \overline{\mathcal{U}}_t f \textit{ for all measurable } f : E \to [0,\infty] \quad (t \geq 0). \tag{31}$$

In particular, if $\underline{\alpha},\overline{\beta}$ are constants and $\underline{\alpha} > 0$, then, for $t > 0$,

$$\overline{\mathcal{U}}_t\infty = \frac{\overline{\beta}}{\underline{\alpha}\,(1 - e^{-\overline{\beta}t})} \quad (\overline{\beta} \neq 0) \quad \textit{and} \quad \overline{\mathcal{U}}_t\infty = \frac{1}{\underline{\alpha}\,t} \quad (\overline{\beta} = 0), \tag{32}$$

and (31) with $f = \infty$ gives

$$P^\mu[\mathcal{X}_t = 0] \geq e^{-\langle\mu,\overline{\mathcal{U}}_t\infty\rangle} \qquad (t > 0). \tag{33}$$

Proof For each $f \in \mathcal{C}_+(E) \cap \mathcal{D}(G)$, the function $\tilde{u}_t := \mathcal{U}_t f$ solves

$$\tfrac{\partial}{\partial t}\tilde{u}_t = G\tilde{u}_t + \beta\tilde{u}_t - \alpha\tilde{u}_t^2 \leq G\tilde{u}_t + \overline{\beta}\tilde{u}_t - \underline{\alpha}\tilde{u}_t^2 \qquad (t \geq 0), \tag{34}$$

and therefore $\mathcal{U}_t f = \tilde{u}_t \leq \overline{\mathcal{U}}_t f$ by Lemma 10. Using Lemmas 8 and 9 this is easily extended to measurable $f : E \to [0,\infty]$, giving (31). Define $\overline{u}$ by the right-hand side of the equations in (32). Then it is easy to check that $\overline{u}$ solves $\frac{\partial}{\partial t}\overline{u}_t = \overline{\beta}\overline{u}_t - \underline{\alpha}\overline{u}_t^2$ $(t > 0)$ with $\lim_{t\to 0}\overline{u}_t = \infty$, and therefore (33) follows from the fact that

$$P^\mu[\mathcal{X}_t = 0] = E^\mu[e^{-\langle \mathcal{X}_t, \infty\rangle}] = e^{-\langle \mu, \mathcal{U}_t \infty\rangle} \qquad (t \geq 0,\ \mu \in \mathcal{M}(E)), \tag{35}$$

and a little approximation argument. ∎

2.2 Some consequences of the finite ancestry property

Let $\mathcal{X}$ be a (G,α,β)-superprocess as in the last section. In line with Lemma 6, we say that $\mathcal{X}$ has the *finite ancestry property* if

$$\inf_{x\in E} P^{\delta_x}[\mathcal{X}_t = 0] > 0 \qquad (t > 0). \tag{36}$$

Note that by (35), property (36) is equivalent to $\|\mathcal{U}_t\infty\|_\infty < \infty$ $(t > 0)$. In this section we prove three simple consequences of the finite ancestry property.

Lemma 12 (Extinction versus explosion) *Assume that the (G,α,β)-superprocess $\mathcal{X}$ has the finite ancestry property. Then, for any $\mu \in \mathcal{M}(E)$,*

$$P^\mu\big[\mathcal{X}_t = 0 \textit{ eventually} \ \ \textit{or} \ \ \lim_{t\to\infty}\langle \mathcal{X}_t, 1\rangle = \infty\big] = 1. \tag{37}$$

Proof We use a general fact about tail events of strong Markov processes, the statement and proof of which can be found in the appendix. Consider the tail event $A := \{\mathcal{X}_t = 0 \text{ eventually}\}$. By Lemma 23 in the appendix

$$\lim_{t\to\infty} P^{\mathcal{X}_t}(A) = 1_A \qquad \text{a.s.} \tag{38}$$

For any fixed $T > 0$, by (35),

$$P^\mu(A) \geq P^\mu[\mathcal{X}_T = 0] = e^{-\langle \mu, \mathcal{U}_T\infty\rangle} \geq e^{-\langle \mu, 1\rangle \|\mathcal{U}_T\infty\|_\infty} \qquad (\mu \in \mathcal{M}(E)). \tag{39}$$

Hence (38) implies that

$$\liminf_{t\to\infty} e^{-\langle \mathcal{X}_t, 1\rangle \|\mathcal{U}_T\infty\|_\infty} \leq 1_A \qquad \text{a.s.} \tag{40}$$

By the finite ancestry property, $\|\mathcal{U}_T\infty\|_\infty < \infty$ and therefore $\lim_{t\to\infty}\langle \mathcal{X}_t, 1\rangle = \infty$ a.s. on $A^{\rm c}$. ∎

The following is a simple consequence of Lemma 12.

Lemma 13 (Extinction of (sub-) critical processes) *Assume that the (G,α,β)-superprocess $\mathcal{X}$ has the finite ancestry property and that $\beta \leq 0$. Then, for any $\mu \in \mathcal{M}(E)$,*

$$P^\mu\big[\mathcal{X}_t = 0 \textit{ eventually}\big] = 1. \tag{41}$$

Proof Since $E^\mu[\langle\mathcal{X}_t,1\rangle]\leq\langle\mu,1\rangle$, $P^\mu[\lim_{t\to\infty}\langle\mathcal{X}_t,1\rangle=\infty]=0$. Now the claim follows from Lemma 12. ∎

Our final result of this section is the following.

Lemma 14 (Extinction versus exponential growth) *Assume that the* (G,α,β)*-superprocess* $\mathcal{X}$ *has the finite ancestry property and that* $\beta>0$ *is a constant. Then, for any* $\mu\in\mathcal{M}(E)$*, there exists a nonnegative random variable* W*, depending on* μ*, such that*

$$\begin{array}{rll} \text{(i)} & \lim_{t\to\infty} e^{-\beta t}\langle\mathcal{X}_t,1\rangle = W & P^\mu\text{-a.s.},\\ \text{(ii)} & \lim_{t\to\infty} E^\mu\big[|e^{-\beta t}\langle\mathcal{X}_t,1\rangle - W|^2\big]=0, & \\ \text{(iii)} & E^\mu(W)=\langle\mu,1\rangle, & \\ \text{(iv)} & \mathrm{Var}^\mu(W)\leq 2\beta^{-1}\|\alpha\|_\infty\langle\mu,1\rangle, & \\ \text{(v)} & \{W=0\}=\{\mathcal{X}_t=0 \textit{ eventually}\} & P^\mu\text{-a.s.} \end{array} \tag{42}$$

Proof Put $\mathcal{V}_tf:=e^{\beta t}S_t$. The mean and covariance of $\mathcal{X}$ are given by the following formulas (see, for example, [Fit88]):

$$\left.\begin{array}{rl} \text{(i)} & E^\mu[\langle\mathcal{X}_t,f\rangle]=\langle\mu,\mathcal{V}_tf\rangle\\ \text{(ii)} & \mathrm{Cov}^\mu(\langle\mathcal{X}_t,f\rangle,\langle\mathcal{X}_t,g\rangle)=2\displaystyle\int_0^t \mathrm{d}s\,\langle\mu,\mathcal{V}_s(\alpha\,(\mathcal{V}_{t-s}f)(\mathcal{V}_{t-s}g))\rangle \end{array}\right\}\quad (t\geq 0,\ f,g\in B(E)). \tag{43}$$

Therefore,

$$E^\mu[\langle\mathcal{X}_t,f\rangle]=e^{\beta t}\langle\mu,S_tf\rangle\qquad(t\geq 0,\ f\in B(E)), \tag{44}$$

and

$$\begin{aligned} \mathrm{Var}^\mu(\langle\mathcal{X}_t,f\rangle) &= 2\int_0^t \mathrm{d}s\, e^{\beta s}e^{2\beta(t-s)}\langle\mu,S_s(\alpha(S_{t-s}f)^2)\rangle\\ &\leq 2\|\alpha\|_\infty\|f\|_\infty^2\langle\mu,1\rangle e^{\beta t}\int_0^t \mathrm{d}s\, e^{\beta(t-s)}\\ &\leq 2\beta^{-1}\|\alpha\|_\infty\|f\|_\infty^2\langle\mu,1\rangle e^{2\beta t}\qquad (t\geq 0,\ f\in B(E)). \end{aligned} \tag{45}$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $\mathcal{X}$ and put

$$\tilde{\mathcal{X}}_t:=e^{-\beta t}\mathcal{X}_t\qquad(t\geq 0). \tag{46}$$

Then (44) and (45) show that for any $0\leq s\leq t$ and $f\in B(E)$

$$\begin{array}{rl} \text{(i)} & E^\mu\big[\langle\tilde{\mathcal{X}}_t,f\rangle\big|\mathcal{F}_s\big]=\langle\tilde{\mathcal{X}}_s,S_{t-s}f\rangle\quad\text{a.s.},\\ \text{(ii)} & \mathrm{Var}^\mu\big[\langle\tilde{\mathcal{X}}_t,f\rangle\big|\mathcal{F}_s\big]\leq 2\beta^{-1}\|\alpha\|_\infty\|f\|_\infty^2\langle\tilde{\mathcal{X}}_s,1\rangle e^{-\beta s}\quad\text{a.s.} \end{array} \tag{47}$$

Since $S_{t-s}1=1$, formula (47 i) shows that $(\langle\tilde{\mathcal{X}}_t,1\rangle)_{t\geq 0}$ is a nonnegative martingale, and hence there exists a nonnegative random variable W such that (42 i) holds. Setting $s=0$ in (47 ii), we see that

$$\mathrm{Var}^\mu\big[\langle\tilde{\mathcal{X}}_t,1\rangle\big]\leq 2\beta^{-1}\|\alpha\|_\infty\langle\mu,1\rangle\qquad(t\geq 0). \tag{48}$$

This implies (42 ii), and, using Fatou, (42 iv). Moreover, by (48) the random variables $\langle\mathcal{X}_t,1\rangle_{t\geq 0}$ are uniformly integrable, and therefore (42 iii) holds.

We are left with the task to prove (42 v). The inclusion $\supset$ is trivial. Formulas (42 iii) and (42 iv) imply that

$$\langle\mu,1\rangle^2 P^\mu[W=0] \leq \mathrm{Var}^\mu(W) \leq 2\beta^{-1}\|\alpha\|_\infty\langle\mu,1\rangle, \tag{49}$$

and therefore

$$P^\mu[W>0] \geq 1-2\beta^{-1}\|\alpha\|_\infty\langle\mu,1\rangle^{-1} \qquad (\mu\neq 0). \tag{50}$$

Note that $\{W>0\}$ is a tail event. Thus, by Lemma 23 in the appendix,

$$\lim_{t\to\infty} P^{\mathcal{X}_t}[W>0] = 1_{\{W>0\}} \quad \text{a.s.} \tag{51}$$

Formula (50) shows that

$$\liminf_{t\to\infty} P^{\mathcal{X}_t}[W>0] \geq 1_{\{\lim_{t\to\infty}\langle\mathcal{X}_t,1\rangle=\infty\}}. \tag{52}$$

Combining Lemma 12 with formulas (51) and (52) we see that $\{\mathcal{X}_t = 0 \text{ eventually}\}^{\mathrm{c}} \subset \{\lim_{t\to\infty}\langle\mathcal{X}_t,1\rangle=\infty\} \subset \{W>0\}$ a.s. ∎

2.3 Smoothness of two log-Laplace semigroups

We return to the special situation $E=[0,1]$ and $G=\overline{A}$ or $G=\overline{A^v}$, where $\overline{A}$ and $\overline{A^v}$ are the closures in $\mathcal{C}(E)$ of the operators A in (5) and A^v in (20), respectively, with domains $\mathcal{D}(A)=\mathcal{D}(A^v):=\mathcal{C}^2[0,1]$, the space of real functions on $[0,1]$ that are twice continuously differentiable. Let $\mathcal{U}=\mathcal{U}(\overline{A},\gamma,\gamma)$ and $\mathcal{U}^v=\mathcal{U}(\overline{A^v},\gamma v,\gamma-1)$ denote the log-Laplace semigroups of the super-Wright-Fisher diffusion $\mathcal{X}$ and the weighted super-Wright-Fisher diffusion $\mathcal{X}^v$, respectively, where $\gamma>0$ is constant. In this section we prove:

Lemma 15 (Smoothing property of $\mathcal{U}$ and $\mathcal{U}^v$) *One has $\mathcal{U}_t(B_+[0,1])\subset\mathcal{C}_+[0,1]$ and $\mathcal{U}^v_t(B_+[0,1])\subset\mathcal{C}_+[0,1]$ for all $t>0$.*

To prepare for the proof, we start with the following elementary property of the semigroups S and S^v generated by $\overline{A}$ and $\overline{A^v}$, respectively (recall (5) and (20)).

Lemma 16 (Strong Feller property) *The semigroups S and S^v have the strong Feller property, i.e., $S_t(B[0,1])\subset\mathcal{C}[0,1]$ and $S^v_t(B[0,1])\subset\mathcal{C}[0,1]$ for all $t>0$.*

Proof Couple two realizations ξ^x,ξ^y of the process with generator $\overline{A}$, started in $x,y\in[0,1]$, in such a way that ξ^x and ξ^y move independently up to the random time $\tau:=\inf\{t\geq 0:\xi^x_t=\xi^y_t\}$, and such that $\xi^x_t=\xi^y_t$ for all $t\geq\tau$. (Here the superscript in ξ^x refers to the initial condition, and not, like elsewhere in this paper, to an h-transform.) Then it is not hard to see that

$$P[\xi^y_t=\xi^x_t]\to 1 \quad \text{as} \quad y\to x \quad \forall t>0. \tag{53}$$

In particular, (53) holds also for $x\in\{0,1\}$ since the boundary is attainable. Since $|S_tf(x)-S_tf(y)|\leq 2\|f\|_\infty P[\xi^x_t\neq\xi^y_t]$, formula (53) shows that $S_tf\in\mathcal{C}[0,1]$ for all $f\in B[0,1]$ and $t>0$. For the process with generator $\overline{A^v}$ the argument is similar but easier, since in this case $\{0,1\}$ is an entrance boundary. ∎

Proof of Lemma 15 Fix $f\in B[0,1]$. The function $u_t:=\mathcal{U}_tf$ is a mild solution of (6), i.e., (see (24))

$$u_t = S_tf+\int_0^t S_{t-s}\big(\gamma u_s(1-u_s)\big)\mathrm{d}s \qquad (t\geq 0), \tag{54}$$

where by Lemma 16, $S_t f$ and $S_{t-s}\big(\gamma u_s(1-u_s)\big)$ are continuous functions on $[0,1]$, for all $0 \leq s < t$. Since the integral is continuous with respect to bounded pointwise convergence of the integrand, we see that $\mathcal{U}_t f(x)$ is continuous in x for all $t > 0$. The same argument applies to $\mathcal{U}^v_t f$. ∎

2.4 Bounds on the absorption probability

Let $\mathcal{U} = \mathcal{U}(\overline{A}, \gamma, \gamma)$. Since the points $0, 1$ are traps for the Wright-Fisher diffusion, $f(r) = 0$ implies $\mathcal{U}_t f(r) = 0$ $(r = 0, 1)$. We have already seen (Lemma 15) that $\mathcal{U}_t f$ is continuous for each $t > 0$. The following lemma shows that if $f(r) = 0$, then $\mathcal{U}_t f$ has a finite slope at $r = 0, 1$, for all $t > 0$. By symmetry, it suffices to consider the case $r = 0$.

Lemma 17 (Absorption of the super-Wright-Fisher diffusion) *Let $\mathcal{U} = \mathcal{U}(\overline{A}, \gamma, \gamma)$, with $\gamma > 0$. Then*

$$\mathcal{U}_t(\infty 1_{(0,1]})(x) \leq K_t\, x \qquad (t > 0,\ x \in [0,1]), \tag{55}$$

with

$$K_t := \frac{e^{\gamma t/2}}{1 - e^{-\gamma t/2}}\Big(\frac{8}{t} + 2\Big) \qquad (t > 0). \tag{56}$$

Note that (55) implies that

$$P^{\delta_x}[\mathcal{X}_t((0,1]) > 0] \leq 1 - e^{-K_t x} \leq K_t\, x \qquad (t > 0,\ x \in [0,1]). \tag{57}$$

We begin with a preparatory lemma.

Lemma 18 (Absorption of the Wright-Fisher diffusion) *For the Wright-Fisher diffusion ξ,*

$$P^x[\xi_t > 0] \leq \Big(\frac{4}{t} + 2\Big)x \qquad (t > 0,\ x \in [0,1]). \tag{58}$$

Proof For $x \geq 0$ put

$$f_0(x) := 1_{\{0\}}(x) \quad \text{and} \quad f_t(x) := (1-2x)e^{-\frac{4x}{t}} 1_{[0,\frac{1}{2}]}(x) \qquad (t > 0). \tag{59}$$

A little calculation shows that for $t > 0$ and $x \geq 0$,

$$\begin{aligned} \tfrac{\partial}{\partial t} f_t(x) &= 4x(1-2x)t^{-2}e^{-\frac{4x}{t}} 1_{[0,\frac{1}{2}]}(x) \\ \tfrac{1}{2}x(1-x)D_x^2 f_t(x) &= \big(8x(1-x)(1-2x)t^{-2}e^{-\frac{4x}{t}} + 8x(1-x)t^{-1}e^{-\frac{4x}{t}}\big)1_{[0,\frac{1}{2}]}(x) \\ &\quad + 2t^{-1}e^{-\frac{2}{t}}\delta_{\frac{1}{2}}(x), \end{aligned} \tag{60}$$

where D_x^2 denotes the generalized second derivative with respect to x and $\delta_{\frac{1}{2}}$ is the delta-function at $\frac{1}{2}$. Since $4x \leq 8x(1-x)$ for all $x \in [0, \frac{1}{2}]$, it follows that

$$\tfrac{\partial}{\partial t} f_t(x) \leq \tfrac{1}{2}x(1-x)D_x^2 f_t(x) \qquad (t > 0,\ x \geq 0). \tag{61}$$

If f_t were contained in $\mathcal{D}(\overline{A})$, then (61) would mean that $\frac{\partial}{\partial t} f_t \leq \overline{A} f_t$ for $t > 0$, and a standard argument (compare Lemma 10) would tell us that $f_t \leq S_t f_0$, where S is the semigroup of ξ. In the present case, we need a little approximation argument.

Let $\phi_n \geq 0$ $(n \geq 0)$ denote $\mathcal{C}^\infty$-functions defined on $[0,\infty)$ with support contained in $[0,\frac{1}{3}]$, say, such that $\phi_n(x)\mathrm{d}x$ are probability measures converging weakly to the δ-measure δ_0 as $n \to \infty$. Put

$$f_t^n(x) := \int_0^\infty \mathrm{d}y\, \phi_n(y) f_t(x+y) =: \phi_n * f_t(x) \qquad (t>0,\ x\geq 0). \tag{62}$$

Then

$$\begin{aligned} \tfrac{\partial}{\partial t} f_t^n(x) &= \phi_n * \tfrac{\partial}{\partial t} f_t(x) \\ \tfrac{\partial^2}{\partial x^2} f_t^n(x) &= \phi_n * D_x^2 f_t(x), \end{aligned} \tag{63}$$

and therefore (61) shows that

$$\tfrac{\partial}{\partial t} f_t^n(x) \leq \tfrac{1}{2}x(1-x)\tfrac{\partial^2}{\partial x^2} f_t^n(x) \qquad (t>0,\ x\geq 0,\ n\geq 0). \tag{64}$$

Since $f_t^n \in \mathcal{D}(\overline{A})$ for all $t>0$, the argument mentioned above gives

$$f_{t+\varepsilon}^n \leq S_t f_\varepsilon^n \qquad (t\geq 0,\ \varepsilon>0). \tag{65}$$

Letting $n\to\infty$ and afterwards $\varepsilon \to 0$ we find that

$$f_t(x) \leq S_t f_0(x) = P^x[\xi_t = 0] \qquad (t\geq 0,\ x\in[0,1]). \tag{66}$$

Note that $\frac{\partial}{\partial x}(1-f_t(x)) = (1-2x)4t^{-1}e^{-\frac{4x}{t}} + 2e^{-\frac{4x}{t}} \leq (\frac{4}{t}+2)$ for $x\in[0,\frac{1}{2}]$. Therefore (66) implies (58). (Note that (58) is trivial for $x \in [\frac{1}{2},1]$.) ∎

Proof of Lemma 17 Fix $f \in B_+[0,1]$ satisfying $f(0)=0$ and write $\mathcal{U}_t f = \mathcal{U}_{t/2}\mathcal{U}_{t/2} f$. By (33) from Lemma 11, $\mathcal{U}_{t/2} f \leq (1-e^{-\gamma t/2})^{-1}$. Since moreover $\mathcal{U}_{t/2} f(0)=0$ because of absorbtion at zero, we have

$$\mathcal{U}_t f \leq \mathcal{U}_{t/2}((1-e^{-\gamma t/2})^{-1} 1_{(0,1]}) \qquad (t>0). \tag{67}$$

Using (31) from Lemma 11, we may estimate $\mathcal{U}(\overline{A},\gamma,\gamma)$ in terms of $\mathcal{U}(\overline{A},0,\gamma)$, which is just the linear semigroup $(e^{\gamma t}S_t)_{t\geq 0}$. Thus, by Lemma 18,

$$\begin{aligned} \mathcal{U}_t f(x) &\leq e^{\gamma t/2} S_{t/2}((1-e^{-\gamma t/2})^{-1} 1_{(0,1]})(x) \\ &\leq e^{\gamma t/2}(1-e^{-\gamma t/2})^{-1}(\tfrac{8}{t}+2)x \qquad (t>0,\ x\in[0,1]). \end{aligned} \tag{68}$$

Letting $f \uparrow \infty$, by monotonicity we arrive at (55). ∎

2.5 The weighted super-Wright-Fisher diffusion

In this section we prove Lemmas 5 and 6. Recall that ξ, ξ^v are the diffusions in $[0,1]$ with generators $\overline{A}, \overline{A^v}$ defined in (5) and (20), and associated semigroups S, S^v, respectively, and that $\mathcal{U} = \mathcal{U}(\overline{A},\gamma,\gamma)$ and $\mathcal{U}^v = \mathcal{U}(\overline{A^v},\gamma v,\gamma-1)$.

Lemma 19 (v-transformed log-Laplace semigroup) *If $f \in \mathcal{D}(\overline{A^v})$, then $vf \in \mathcal{D}(\overline{A})$ and*

$$\overline{A}(vf) = v\,(\overline{A^v}-1)f. \tag{69}$$

Moreover,

$$\mathcal{U}_t(vf) = v\,\mathcal{U}_t^v f \qquad (t\geq 0,\ f\in B_+[0,1]). \tag{70}$$

Proof For any $f \in \mathcal{C}^2[0,1]$, it is easy to check that

$$A(vf) = v\,(A^v - 1)f. \tag{71}$$

Fix $f \in \mathcal{D}(\overline{A^v})$ and choose $f_n \in \mathcal{C}^2[0,1]$ such that $f_n \to f$ in $\mathcal{C}[0,1]$. Then (71) shows that $A(vf_n) \to v\,(\overline{A^v} - 1)f$, which implies that $vf \in \mathcal{D}(\overline{A})$ and that (69) holds.

Now fix $f \in \mathcal{C}_+[0,1] \cap \mathcal{D}(\overline{A^v})$ and put $u^v_t := \mathcal{U}^v_t f$ $(t \geq 0)$. Then u^v is the classical solution of the Cauchy equation

$$\begin{cases} \frac{\partial}{\partial t} u^v_t = \overline{A^v} u^v_t + (\gamma - 1) u^v_t - \gamma v\,(u^v_t)^2 & (t \geq 0), \\ u^v_0 = f. \end{cases} \tag{72}$$

It follows from (69) that

$$\begin{aligned} \frac{\partial}{\partial t} v u^v_t &= v \frac{\partial}{\partial t} u^v_t = v\overline{A^v} u^v_t + (\gamma - 1) v u^v_t - \gamma\,(v u^v_t)^2 \\ &= \overline{A}(v u^v_t) + \gamma v u^v_t - \gamma\,(v u^v_t)^2 \qquad (t \geq 0), \end{aligned} \tag{73}$$

i.e., $u_t := v u^v_t$ is the classical solution to the Cauchy equation

$$\begin{cases} \frac{\partial}{\partial t} u_t = \overline{A} u_t + \gamma u_t - \gamma u_t^2 & (t \geq 0), \\ u_0 = vf. \end{cases} \tag{74}$$

This proves that $\mathcal{U}_t(vf) = u_t = v u^v_t = v\mathcal{U}^v_t f$ for all $f \in \mathcal{C}_+[0,1] \cap \mathcal{D}(\overline{A^v})$. The general case follows from Lemma 8 and the fact that the class of $f \in B_+[0,1]$ for which (70) holds is closed under bounded pointwise limits. ∎

Proof of Lemma 5 Set $\mathcal{F}_t := \sigma(\mathcal{X}_s : 0 \leq s \leq t)$. Then by (70), for all $0 \leq s \leq t$ and $f \in B_+[0,1]$,

$$\begin{aligned} &E\big[e^{-\langle v\mathcal{X}_t, f\rangle} \big| \mathcal{F}_s\big] = E\big[e^{-\langle \mathcal{X}_t, vf\rangle} \big| \mathcal{F}_s\big] = e^{-\langle \mathcal{X}_s, \mathcal{U}_{t-s}(vf)\rangle} \\ &= e^{-\langle \mathcal{X}_s, v\mathcal{U}^v_{t-s} f\rangle} = e^{-\langle v\mathcal{X}_s, \mathcal{U}^v_{t-s} f\rangle}. \end{aligned} \tag{75}$$

It follows that $(v\mathcal{X}_t)_{t \geq 0}$ is a Markov process and that its transition probabilities coincide with those of the $(\overline{A^v}, \gamma v, \gamma - 1)$-superprocess. Since $\mathcal{X}$ has continuous sample paths, so has $v\mathcal{X}$. ∎

Proof of Lemma 6 We need to prove (22), which by (35) is equivalent to the statement that $\|\mathcal{U}^v_t \infty\|_\infty < \infty$ for all $t > 0$. Assume that $f \in B_+[0,1]$ satisfies $f(0) = f(1) = 0$. By Lemma 17, $\mathcal{U}_t f(x) \leq K_t\, x$ for the constant K_t mentioned there. By symmetry, one also has $\mathcal{U}_t f(x) \leq K_t\,(1 - x)$ and, since $x \wedge (1 - x) \leq \frac{1}{3} v(x)$, $\mathcal{U}_t f(x) \leq \frac{1}{3} K_t\, v(x)$. Let $g \in B_+[0,1]$. By formula (70) and the fact that $(vg)(0) = (vg)(1) = 0$, we see that $\mathcal{U}^v_t g(x) = \frac{1}{v(x)} \mathcal{U}_t(vg)(x) \leq \frac{1}{3} K_t$ for all $x \in (0,1)$. By Lemma 16, $\mathcal{U}^v_t g$ is continuous on $[0,1]$ and therefore $\mathcal{U}^v_t g(x) \leq \frac{1}{3} K_t$ holds also for $x = 0, 1$. Taking the limit $g \uparrow \infty$ we see that $\|\mathcal{U}^v_t \infty\|_\infty \leq \frac{1}{3} K_t < \infty$ for all $t > 0$. ∎

2.6 Ergodicity of the v-transformed Wright-Fisher diffusion

Recall that ξ^v is the diffusion on $[0,1]$ with generator $\overline{A^v}$ defined in (20) and associated semigroup S^v. As in Theorem 1, ℓ denotes the Lebesgue measure on $(0,1)$ and v is defined by (9). In this section we prove:

Lemma 20 (Ergodicity of the v-transformed Wright-Fisher diffusion) *The Markov process ξ^v has the unique invariant law $v\ell$ and is ergodic:*

$$\lim_{t\to\infty}\|S^v_t f-\langle v\ell,f\rangle\|_\infty=0\qquad\forall f\in B[0,1].\tag{76}$$

Proof Since

$$\tfrac{\partial}{\partial x}\big[\tfrac{1}{2}x(1-x)v(x)\big]=2(\tfrac{1}{2}-x)v(x)\qquad(x\in[0,1]),\tag{77}$$

$v\ell$ is a (reversible) invariant law for the process with generator $\overline{A^v}$ (see [EK86, Proposition 4.9.2]). Fix $x\in[0,1]$. Let ξ^v be the process started in x and let $\tilde\xi^v$ be the process started in the invariant law $v\ell$. Then $\xi^v,\tilde\xi^v$ may represented as solutions to the SDE

$$\mathrm{d}\xi^v_t=2(\tfrac{1}{2}-\xi^v_t)\mathrm{d}t+\sqrt{\xi^v_t(1-\xi^v_t)}\mathrm{d}B_t,\tag{78}$$

relative to the same Brownian motion B. Using the technique of Yamada & Watanabe (see, for example [EK86, Theorem 5.3.8]), it is easy to prove that

$$E[|\xi^v_t-\tilde\xi^v_t|]=e^{-2t}E[|\xi^v_0-\tilde\xi^v_0|]\le e^{-2t}\qquad(t\ge 0).\tag{79}$$

It follows that for any function f satisfying $|f(y)-f(z)|\le|y-z|\quad(y,z\in[0,1])$,

$$\big|E[f(\xi^v_t)]-\langle v\ell,f\rangle\big|\le E[|f(\xi^v_t)-f(\tilde\xi^v_t)|]\le e^{-2t}.\tag{80}$$

This implies that the function $x\mapsto\mathcal{L}^x(\xi^v_t)$ from $[0,1]$ into the space $\mathcal{M}_1[0,1]$ of probability measures on $[0,1]$, converges as $t\to\infty$ uniformly to the constant function $v\ell$. This shows that (76) holds for all $f\in\mathcal{C}[0,1]$. Since ξ^v has the strong Feller property (Lemma 16), (76) holds for all $f\in B[0,1]$. ∎

2.7 Long-time behavior of the weighted super-Wright-Fisher diffusion

The following lemma prepares for the proof of formula (15) in Theorem 1.

Lemma 21 (Mean square convergence) *Assume that $\gamma>1$. Let $\mathcal{X}^v$ be the $(\overline{A^v},\gamma v,\gamma-1)$-superprocess started in $\mathcal{X}^v_0=\mu\in\mathcal{M}[0,1]$. Then there exists a nonnegative random variable W, depending on μ, such that*

$$\begin{array}{ll}\text{(i)} & \lim_{t\to\infty}e^{-(\gamma-1)t}\langle\mathcal{X}^v_t,1\rangle=W\quad\text{a.s.}\\ \text{(ii)} & \lim_{t\to\infty}E^\mu\Big[\big|e^{-(\gamma-1)t}\langle\mathcal{X}^v_t,f\rangle-W\langle v\ell,f\rangle\big|^2\Big]=0\quad\forall f\in B[0,1].\end{array}\tag{81}$$

Moreover,

$$E^\mu(W)=\langle\mu,1\rangle\quad\textit{and}\quad\mathrm{Var}^\mu(W)\le 3\tfrac{\gamma}{\gamma-1}\langle\mu,1\rangle,\tag{82}$$

and

$$\{W=0\}=\{\mathcal{X}^v_t=0\ \textit{eventually}\}\qquad\text{a.s.}\tag{83}$$

Proof Except for formula (81 ii), all statements are direct consequences of the fact that $\mathcal{X}^v$ has the finite ancestry property (Lemma 6) and of Lemma 14 (note that $\|\gamma v\|_\infty=\frac{3}{2}\gamma$).

Fix $f\in B[0,1]$. Let $(\mathcal{F}_t)_{t\ge0}$ be the filtration generated by $\mathcal{X}^v$ and put $\tilde{\mathcal{X}}^v_t:=e^{-(\gamma-1)t}\mathcal{X}^v_t$ $(t\ge 0)$. Pick $1\le s_n\le t_n$ such that $s_n\to\infty$ and $t_n-s_n\to\infty$. Then, by (47),

$$E^\mu\Big[\big|\langle\tilde{\mathcal{X}}^v_{t_n},f\rangle-\langle\tilde{\mathcal{X}}^v_{s_n},S^v_{t_n-s_n}f\rangle\big|^2\Big|\mathcal{F}_{s_n}\Big]\le 3\tfrac{\gamma}{\gamma-1}\|f\|^2_\infty\langle\tilde{\mathcal{X}}^v_{s_n},1\rangle e^{-(\gamma-1)s_n}\quad\text{a.s.}\tag{84}$$

Taking expectations on both sides in (84), one finds that

$$E^\mu\Big[\big|\langle\tilde{\mathcal{X}}^v_{t_n},f\rangle-\langle\tilde{\mathcal{X}}^v_{s_n},S^v_{t_n-s_n}f\rangle\big|^2\Big]\leq 3\tfrac{\gamma}{\gamma-1}\|f\|^2_\infty\langle\mu,1\rangle e^{-(\gamma-1)s_n}. \tag{85}$$

By (42 ii),

$$\lim_{t\to\infty}E^\mu\big[|\langle\tilde{\mathcal{X}}^v_t,1\rangle-W|^2\big]=0. \tag{86}$$

Using Lemma 20 (about the ergodicity of ξ^v) and (86), it is easy to show that

$$\lim_{n\to\infty}E^\mu\Big[\big|\langle\tilde{\mathcal{X}}^v_{s_n},S^v_{t_n-s_n}f\rangle-W\langle v\ell,f\rangle\big|^2\Big]=0. \tag{87}$$

Combining this with (85), we see that

$$\lim_{n\to\infty}E^\mu\Big[\big|\langle\tilde{\mathcal{X}}^v_{t_n},f\rangle-W\langle v\ell,f\rangle\big|^2\Big]=0. \tag{88}$$

Since this is true for any $t_n\to\infty$, (81 ii) follows. ∎

2.8 Long-time behavior of the super-Wright-Fisher diffusion

Proof of Theorem 1 Using Lemma 5, we can translate our results on the weighted super-Wright-Fisher diffusion $\mathcal{X}^v$ to the super-Wright-Fisher diffusion $\mathcal{X}$. Thus, Lemma 21 proves formulas (10 ii), (11 ii), and (14)–(15), where $W_{(0,1)}$ is the random variable W from Lemma 21. Formula (13) follows from Lemma 13. To finish the proof of Theorem 1, it suffices to prove (10 i), (11 i) and (12).

1. Proof of formula (10 i) One has $E^\mu[\langle\mathcal{X}_t,f\rangle]=e^{\gamma t}\langle\mu,S_tf\rangle$ for all $t\geq 0$, $f\in B[0,1]$ by (44). Since the points $r=0,1$ are traps for the Wright-Fisher diffusion, $E^\mu[\langle\mathcal{X}_t,1_{\{r\}}\rangle]=e^{\gamma t}\langle\mu,S_t1_{\{r\}}\rangle\geq e^{\gamma t}\langle\mu,1_{\{r\}}\rangle$ for all $t\geq 0$, $r=0,1$. Thus, the processes $(e^{-\gamma t}\langle\mathcal{X}_t,1_{\{r\}}\rangle)_{t\geq 0}$ ($r=0,1$) are nonnegative submartingales, and hence there exist random variables W_r ($r=0,1$) such that (10 i) holds.

2. Proof of formula (12) For $\gamma\leq 1$ the statement is trivial by (13), so assume $\gamma>1$. By symmetry it suffices to consider the case $r=0$. From the L_2-convergence formula (15) we have, for any $K>0$,

$$\{W_{(0,1)}>0\}\subset\big\{\forall T<\infty\ \exists t\geq T\text{ such that }\mathcal{X}_t([\tfrac14,\tfrac13])\geq K\big\}\quad\text{a.s.} \tag{89}$$

Assume for the moment that for some $t>0$ and (sufficiently large) K,

$$\inf_{\mu:\,\mu([\frac14,\frac13])\geq K}P^\mu[W_0>0]>0. \tag{90}$$

Then we see from (89) and (90) that

$$\{W_{(0,1)}>0\}\subset\big\{\lim_{t\to\infty}P^{\mathcal{X}_t}[W_0>0]=0\big\}^{\rm c}\subset\{W_0>0\}\quad\text{a.s.}, \tag{91}$$

where the second inclusion follows from the fact that, by Lemma 23 in the appendix,

$$\lim_{t\to\infty}P^{\mathcal{X}_t}[W_0>0]=1_{\{W_0>0\}}\quad\text{a.s.} \tag{92}$$

Thus, we are done if we can prove (90). By the branching property, it suffices to prove (90) for measures μ that are concentrated on $[\frac{1}{4},\frac{1}{3}]$. Fix any $t>0$. Formulas (44) and (45) give

$$\begin{aligned}&\text{(i)}\quad E^\mu\big[\langle\mathcal{X}_t,1_{\{0\}}\rangle\big]=\langle\mu,S_t1_{\{0\}}\rangle e^{\gamma t},\\&\text{(ii)}\quad \mathrm{Var}^\mu\big[\langle\mathcal{X}_t,1_{\{0\}}\rangle\big]\leq 2\langle\mu,1\rangle e^{2\gamma t}.\end{aligned}\tag{93}$$

It follows from formula (66) (recall (59)) that

$$\inf_{x\in[\frac{1}{4},\frac{1}{3}]} S_t1_{\{0\}}(x)>0.\tag{94}$$

Denoting the infimum by ε, we get the bounds

$$\begin{aligned}&\text{(i)}\quad E^\mu\big[\langle\mathcal{X}_t,1_{\{0\}}\rangle\big]\geq\varepsilon\langle\mu,1\rangle e^{\gamma t},\\&\text{(ii)}\quad \mathrm{Var}^\mu\big[\langle\mathcal{X}_t,1_{\{0\}}\rangle\big]\leq 2\langle\mu,1\rangle e^{2\gamma t}.\end{aligned}\tag{95}$$

These formulas show that for large $\langle\mu,1\rangle$, the standard deviation of $\langle\mathcal{X}_t,1_{\{0\}}\rangle$ is small compared to its mean. Therefore, using Chebyshev's inequality, it is easy to show that for every $M>0$ there exists a $K>0$ such that

$$\inf_{\mu\in\mathcal{M}[\frac{1}{4},\frac{1}{3}]:\,\langle\mu,1\rangle\geq K} P^\mu[\langle\mathcal{X}_t,1_{\{0\}}\rangle\geq M]>0.\tag{96}$$

Hence, by the Markov property, in order to prove (90) it suffices to show that for M sufficiently large,

$$\inf_{\mu:\,\mu(\{0\})\geq M} P^\mu[W_0>0]>0.\tag{97}$$

By the branching property, it suffices to prove (97) for measures μ that are concentrated on $\{0\}$. In that case, $\mathcal{X}_t(\{0\})_{t\geq0}$ is an autonomous supercritical Feller's branching diffusion (a superprocess in a single-point space is just a Feller's branching diffusion). Applying Lemma 14 to this Feller's branching diffusion, again using Chebyshev, it is not hard to prove (97). Since the arguments are very similar to those we have already seen, we skip the details.

3. Proof of formula (11 i) The inclusion $\{W_r=0\}\supset\{\mathcal{X}_t(\{r\})=0\text{ eventually}\}$ a.s. is trivial. By (12) and (11 ii), $\{W_r=0\}\subset\{W_{(0,1)}=0\}\subset\{\mathcal{X}_t((0,1))=0\text{ eventually}\}$ a.s. Therefore, by the strong Markov property, it suffices to prove $\{W_r=0\}\subset\{\mathcal{X}_t(\{r\})=0\text{ eventually}\}$ a.s. for the process started in μ with $\mu((0,1))=0$. In this case, $(\mathcal{X}_t(\{r\}))_{t\geq0}$ is an autonomous supercritical Feller's branching diffusion, and the statement is easy (see the previous parapraph). ∎

2.9 Long-time behavior of the log-Laplace semigroup

Proof of Proposition 2 By formula (2),

$$e^{-\langle\mu,\mathcal{U}_tf\rangle}=E^\mu\big[e^{-f(0)\mathcal{X}_t(\{0\})}e^{-f(1)\mathcal{X}_t(\{1\})}e^{-\langle\mathcal{X}_t,1_{(0,1)}f\rangle}\big].\tag{98}$$

By (10 i) and (11 i) in Theorem 1,

$$\lim_{t\to\infty} e^{-f(r)\mathcal{X}_t(\{r\})}=1_{\{f(r)=0\text{ or }W_r=0\}}\quad\text{a.s.}\quad(r=0,1).\tag{99}$$

Now, if $\langle \ell, f \rangle = 0$ for some $f \in B_+[0,1]$, then $e^{-\langle \mathcal{X}_t, 1_{(0,1)} f \rangle} = 1$ a.s. for each $t > 0$. To see this, note that by (43), $E^{\delta_x}[\langle \mathcal{X}_t, 1_{(0,1)} f \rangle] = e^{\gamma t} \langle \delta_x, S_t 1_{(0,1)} f \rangle = e^{\gamma t} E^x[1_{(0,1)}(\xi_t) f(\xi_t)]$ where ξ is the Wright-Fisher diffusion. Since the law of the Wright-Fisher diffusion at any time $t > 0$ (started in an arbitrary initial condition) on $(0,1)$ is absolutely continuous with respect to Lebesgue measure, we see that $E^{\delta_x}[\langle \mathcal{X}_t, 1_{(0,1)} f \rangle] = 0$ and hence $\langle \mathcal{X}_t, 1_{(0,1)} f \rangle = 0$ P^{δ_x}-a.s. (Actually, since $\mathcal{X}$ is a one-dimensional superprocess, it is presumably true that $\mathcal{X}_t$, restricted to $(0,1)$, for $t > 0$ is almost surely absolutely continuous with respect to Lebesgue measure.)

On the other hand, if $\langle \ell, f \rangle > 0$, then by formulas (10 ii), (11 ii), (13), and (15) in Theorem 1,

$$e^{-\langle \mathcal{X}_t, 1_{(0,1)} f \rangle} \stackrel{\mathrm{P}}{\longrightarrow} 1_{\{W_{(0,1)}=0\}}. \tag{100}$$

Hence, for general $f \in B_+[0,1]$

$$e^{-\langle \mathcal{X}_t, 1_{(0,1)} f \rangle} \stackrel{\mathrm{P}}{\longrightarrow} 1_{\{\langle \ell, f \rangle = 0 \text{ or } W_{(0,1)}=0\}}, \tag{101}$$

where $\stackrel{\mathrm{P}}{\longrightarrow}$ denotes convergence in probability. Inserting (99) and (101) into (98) we arrive at the first equality in (16). Using formula (12) and checking the eight possibilities for $f(0), f(1), \langle \ell, f \rangle$ to be zero or positive, we find the second equality in (16). ∎

2.10 Long-time behavior of binary splitting Wright-Fisher diffusions

Proof of Proposition 4 By Proposition 2, for the functions $p_1, \ldots, p_5$ from (17),

$$\left.\begin{array}{l} p_1(x) = 0, \\ p_2(x) = \lim_{t\to\infty} \mathcal{U}_t 1_{(0,1)}(x), \\ p_3(x) = \lim_{t\to\infty} \mathcal{U}_t 1_{\{0\}}(x) = \lim_{t\to\infty} \mathcal{U}_t 1_{[0,1)}(x), \\ p_4(x) = \lim_{t\to\infty} \mathcal{U}_t 1_{\{1\}}(x) = \lim_{t\to\infty} \mathcal{U}_t 1_{(0,1]}(x), \\ p_5(x) = \lim_{t\to\infty} \mathcal{U}_t 1 \end{array}\right\} \quad (x \in [0,1]). \tag{102}$$

Since by formula (4), for each Borel measurable $B \subset [0,1]$, $P^{\delta_x}[X_t(B) > 0] = U_t 1_B = \mathcal{U}_t 1_B(x)$ $(t \geq 0,\ x \in [0,1])$, we can rewrite the expressions in the right-hand side of (102) as in (18). ∎

2.11 Smoothness of fixed points

In order to finish the proof of Proposition 3 we need to show that the functions $p_1, \ldots, p_5$ occurring there are twice continuously differentiable on $[0,1]$. We begin with the following.

Lemma 22 (Smoothness of fixed points) *If $p \in B_+[0,1]$ is a fixed point under $\mathcal{U}(\overline{A}, \gamma, \gamma)$, then $p \in \mathcal{D}(\overline{A})$ and $\overline{A}p + \gamma p(1-p) = 0$.*

Proof For any $t \geq 0$, Lemma 15 implies that $p = \mathcal{U}_t p \in \mathcal{C}_+[0,1]$. Moreover, since $u_t := p$ $(t \geq 0)$ is a mild solution of (6) (recall (54)),

$$p = S_t p + \int_0^t S_s\big(\gamma p(1-p)\big)\mathrm{d}s \qquad (t \geq 0). \tag{103}$$

Hence

$$\overline{A}p := \lim_{t\to 0} t^{-1}(S_t p - p) = -\lim_{t\to 0} t^{-1} \int_0^t S_s\big(\gamma\, p(1-p)\big)\mathrm{d}s = -\gamma\, p(1-p), \tag{104}$$

where the limit exists in $\mathcal{C}[0,1]$. ∎

In this one-dimensional situation, the domain of $\overline{A}$ is known explicitly. One has (see [EK86, Theorem 8.1.1])

$$\mathcal{D}(\overline{A}) = \Big\{ f \in \mathcal{C}[0,1] \cap \mathcal{C}^2(0,1) : \lim_{x \to r} \tfrac{1}{2} x(1-x) \tfrac{\partial^2}{\partial x^2} f(x) = 0 \quad (r = 0,1) \Big\}. \tag{105}$$

Here $\mathcal{C}[0,1] \cap \mathcal{C}^2(0,1)$ denotes the class of continuous real functions on $[0,1]$ that are twice continuously differentiable on $(0,1)$.

Proof of Proposition 3 We only have to prove the smoothness statement, all other statements having been proved in the text. It suffices to show that p_2 and p_4 are twice continuously differentiable on $[0,1]$ and solve (8). The statement for p_3 then follows by symmetry, while for $p_1 = 0$ and $p_5 = 1$ (see Proposition 4), the claim is obvious. Since p_2, p_4 are fixed points under $\mathcal{U}(\overline{A}, \gamma, \gamma)$, it follows from Lemma 22 and formula (105) that p_2, p_4 are continuous on $[0,1]$, twice continuously differentiable on $(0,1)$, and solve equation (8) on $(0,1)$. We are done if we can show that their first and second derivatives can be extended to continuous functions on $[0,1]$. (If f is twice continuously differentiable on $(0,1)$ and the limits $\lim_{x \to r} \frac{\partial}{\partial x} f(x)$ and $\lim_{x \to r} \frac{\partial^2}{\partial x^2} f(x)$ exists $(r = 0,1)$, then these limits coincide with the one-sided derivatives on the boundary. This follows, for example, from Corollary 6.3 in the appendix of [EK86].)

Proposition 4 shows that $p_2, p_4 \leq 1$ and therefore, since they solve (8) on $(0,1)$, p_2 and p_4 are concave. Proposition 4 also shows that $p_2(0) = p_2(1) = 0$ and $p_4(0) = 0$, $p_4(1) = 1$. (See Figure 2 as an illustration.) Since p_2 is concave, $\frac{\partial}{\partial x} p_2(x)$ increases to a limit in $(-\infty, \infty]$ as $x \downarrow 0$. Lemma 17 implies that this limit is finite, and therefore $\frac{\partial}{\partial x} p_2(x)$ is continuous at $x = 0$. Since p_2 solves (8) on $(0,1)$,

$$\lim_{x \to 0} \tfrac{\partial^2}{\partial x^2} p_2(x) = \lim_{x \to 0} \frac{2\gamma p_2(x)(1 - p_2(x))}{x(1-x)} = 2\gamma \tfrac{\partial}{\partial x} p_2(x) \big|_{x=0}, \tag{106}$$

which proves that $\frac{\partial^2}{\partial x^2} p_2(x)$ is continuous at $x = 0$. The same argument proves that $\frac{\partial}{\partial x} p_2(x)$ and $\frac{\partial^2}{\partial x^2} p_2(x)$ are continuous at $x = 1$, and that $\frac{\partial}{\partial x} p_4(x)$ and $\frac{\partial^2}{\partial x^2} p_4(x)$ are continuous at $x = 0$. Since p_4 is concave, $\frac{\partial}{\partial x} p_4(x)$ decreases to a limit in $[-\infty, \infty)$ as $x \uparrow 1$. Since $p_4(1) = 1$ and $p_4 \leq 1$, $\frac{\partial}{\partial x} p_4(x) \big|_{x=1} \geq 0$. Since p_4 solves (8) on $(0,1)$ and $\frac{\partial}{\partial x} [p_4(x)(1 - p_4(x))] \big|_{x=1} = -\frac{\partial}{\partial x} p_4(x) \big|_{x=1}$,

$$\lim_{x \uparrow 1} \tfrac{\partial^2}{\partial x^2} p_4(x) = \lim_{x \uparrow 1} \frac{2\gamma p_4(x)(1 - p_4(x))}{x(1-x)} = 2\gamma \tfrac{\partial}{\partial x} p_4(x) \big|_{x=1}, \tag{107}$$

which proves that $\frac{\partial}{\partial x} p_4(x)$ and $\frac{\partial^2}{\partial x^2} p_4(x)$ are continuous at $x = 1$. ∎

Appendix: a zero-one law for Markov processes

Let E be a Polish space and let $(P^x)^{x \in E}$ be a family of probability measures on $\mathcal{D}_E[0, \infty)$ (the space of cadlag functions $w : [0, \infty) \to E$) such that under $(P^x)^{x \in E}$, the coordinate projections $\{w \mapsto w_t =: \xi_t(w) : t \geq 0\}$ form a Borel right process in the sense of [Sha88]. This is true, for example, if $(P^x)^{x \in E}$ are the laws of a Feller process on a locally compact Polish space, or a (G, α, β)-superprocess as introduced in Section 2.1 (see [Fit88]). Let $\mathcal{T} := \bigcap_{t \geq 0} \sigma(\xi_s : s \geq t)$ denote the tail-σ-field of ξ. Let $(\theta_t w)_s := w_{t+s}$ $(t, s \geq 0)$ be the time-shift on $\mathcal{D}_E[0, \infty)$. Then the following holds.

Lemma 23 (Zero-one law for Markov processes) *Assume that $A \in \mathcal{T}$. Then for each $x \in E$,*

$$\lim_{t\to\infty} P^{\xi_t}(\theta_t^{-1}(A)) = 1_A \quad P^x\text{–a.s.} \tag{108}$$

Proof Let $\mathcal{F}_t := \sigma(\xi_s : 0 \leq s \leq t)$ $(t \geq 0)$ be the filtration generated by ξ and set $\mathcal{F}_\infty := \sigma(\xi_s : s \geq 0)$. Since ξ is a Markov process, $P^{\xi_t}(\theta_t^{-1}(A)) = P[A|\mathcal{F}_t]$ a.s. For any sequence of times $t_n \uparrow \infty$ one has $\mathcal{F}_{t_n} \uparrow \mathcal{F}_\infty$ and therefore $P[A|\mathcal{F}_{t_n}] \to P[A|\mathcal{F}_\infty] = 1_A$ a.s. (see [Loe63, § 29, Complement 10 (b)]). Since ξ is a right process, the function $t \mapsto P^{\xi_t}(\theta_t^{-1}(A))$ is a.s. right-continuous (see [Sha88] (7.4.viii)), and we conclude that (108) holds. ∎

Acknowledgments

Work sponsored by the DFG. The authors thank Janos Engländer for answering our questions about his work and Jan Seidler for pointing out the usefulness of the strong Feller property. Anita Winter and Achim Klenke are thanked for useful discussions.

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