Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

On the computation of hyperbolic sets and their invariant manifolds

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submitted: 18th October 1993

[1] This paper was written while the author was visiting the Institute of Applied Analysis and Stochastics.

Preprint No. 68
Berlin 1993

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Ale Jan Homburg

Abstract

We describe a method for finding periodic orbits contained in a hyperbolic invariant set and of constructing their local stable and unstable manifolds, suitable to implement in a computer.

Introduction

In this paper we describe a method for computing (local) stable and unstable manifolds of periodic points of endomorphisms in $\mathbb{R}^n$. Our method is an adaptation of a construction used to prove the stable manifold theorem. We work with contractions on a space of sequences, such that orbits in the local stable or unstable manifold are fixed points of these contractions. This implies that we have good estimates for the error with which points on the local stable or unstable manifold are computed. By iteration under f resp. f^{-1} (if it is known) we get larger parts of the unstable resp. stable manifold.

In the following section we describe these contractions and review there rôle in the proof of the stable manifold theorem. Additional information can be found in [HOV,1993]. Section 2 briefly explores the construction of some normally hyperbolic invariant manifolds, such as strong stable manifolds. In section 3, we consider the problem of finding (periodic) orbits in a hyperbolic invariant set. We work out the theory of the first sections in section 4 for a map possessing a horseshoe, in this case we can also easily approximate orbits in the invariant set with a given symbolic coding. In the final section we provide two examples of numerical computations done for the Henon map and for an endomorphism in $\mathbb{R}^3$.

Some methods to compute local (un)stable manifolds are already available. Nusse and Yorke [NY,1988] have given a nice construction, which is however restricted to codimension one local (un)stable manifolds. Let us give a brief description of their method to compute a local stable manifold for a hyperbolic fixed point of a map in $\mathbb{R}^2$. Assume the origin 0 is a hyperbolic fixed point of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, such that $Df(0)$ has a stable and an unstable eigenvalue, and assume coordinates in which the x-axis is the stable manifold of $Df(0)$ and the y-axis is the unstable manifold of $Df(0)$. Consider a section S close to the origin and transversally intersecting the x-axis. On S there is one point on the local stable manifold of f. Other points on S leave a neighbourhood of the origin when iterating a possibly great number of times. In case the unstable eigenvalue of $Df(0)$ is positive, points stay on the same side of the local stable manifold under iteration by f (as long as the points are in a neighbourhood of the origin). In case the unstable eigenvalue is negative, we replace f by f^2 to obtain a map whose derivative in 0 has a positive unstable eigenvalue. By iterating a point on the section S we can thus see whether this point lies above or below the local stable manifold. Subsequently, a bisection method yields a good approximation of the point on the local stable manifold. This way an approximation of the local stable manifold is obtained, also in more then two dimensions suitable of finding codimension one local stable manifolds. Nusse and Yorke use a similar method to approximate orbits in invariant hyperbolic (not attracting) sets with one unstable direction. With our method it is possible to approximate invariant manifolds with arbitrary codimension.

acknowledgements. In writing this paper, stimulating discussions I had with Floris Takens were very helpfull. I acknowledge support by the Deutsche Forschungsgemeinschaft (DFG) under grant La 525/4-1.

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1 Stable and unstable manifolds

In this section we give the construction used in proving the stable manifold theorem. This construction is a variant of Perron's construction, see [Perron,1929], [Irwin,1980], [Shub,1980]. First we recall some definitions, mostly to fix the notation, see [Shub,1980] for details.

We start by treating diffeomorphisms, endomorphisms will be studied afterwards. Let f be a diffeomorphism on $\mathbb{R}^n$. The stable set $W^s(x)$ is the set of points y such that the distance between $f^n(y)$ and $f^n(x)$ goes to zero for $n \to \infty$. If we want to stress the dependence of the stable set on the map f, we write $W^s_f(x)$.

The local stable set $W^{s,\varepsilon}(x)$, or $W_f^{s,\varepsilon}(x)$, consists of the points y on $W^s(x)$ such that $f^n(y)$ is in an ε-neighbourhood of $f^n(x)$ for all positive n. Observe that

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^{s,\varepsilon}(x)).$$

The unstable set $W^u(x)$ and the local unstable set $W^{u,\varepsilon}(x)$ are defined as the (local) stable set for f^{-1}.

Let $\mathfrak{D} = f(\mathfrak{D})$ be a compact hyperbolic invariant set. The set $\mathfrak{D}$ being hyperbolic means that there is a continuous Df-invariant splitting

$$T_{\mathfrak{D}}\mathbb{R}^n = E^s \oplus E^u,$$

such that for some $m > 0$, $Df^m|_{E^s}$ contracts vectors and $Df^m|_{E^u}$ expands vectors. Through a change of the metric on $\mathbb{R}^n$, we may assume $m = 1$, see [Shub,1980].

We make the identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$, for each $x \in \mathbb{R}^n$. For $x \in \mathfrak{D}$, we write $E^s(x) \oplus E^u(x)$ for the induced splitting of $\mathbb{R}^n$. We write e.g. $E^s_f(x)$ if we want to stress the dependence on f. Let $\pi_s(x) : \mathbb{R}^n \to E^s(x)$ and $\pi_u(x) : \mathbb{R}^n \to E^u(x)$ be the canonical projections defined by the splitting $\mathbb{R}^n = E^s(x) \oplus E^u(x)$, so e.g. the kernel of $\pi_s(x)$ is $E^u(x)$. The projections $\pi_s(x)$ and $\pi_u(x)$ depend continuously on x and satisfy

$$Df(x)\pi_s(x) = \pi_s(f(x)), Df(x)\pi_u(x) = \pi_u(f(x)).$$

We can make a continuous extension of these projections to a neighbourhood of $\mathfrak{D}$. We write $y = (\pi_s(x)y, \pi_u(x)y)$, where the projections are centred in 0, not in x.

The stable manifold theorem now states that the stable set of a point in $\mathfrak{D}$ is an injectively immersed manifold, as smooth as f, see [Shub,1980], [Irwin,1980].

Theorem 1.1 *Let $\mathfrak{D}$ be a compact hyperbolic invariant set for a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. Then the stable set $W^s(x)$ and the unstable set $W^u(x)$ of a point $x \in \mathfrak{D}$, are injectively immersed manifolds, as smooth as f.*

SKETCH OF PROOF. Let $\{x_k\}_{k\in\mathbb{Z}}$ be an orbit in $\mathfrak{D}$, meaning $x_{k+1} = f(x_k) \in \mathfrak{D}$. Consider the set $C_b(\mathbb{N}, \mathbb{R}^n)$ of bounded sequences $\mathbb{N} \to \mathbb{R}^n$, equipped with the supnorm.

Consider the following map Γ_f on $C_b(\mathbb{N}, \mathbb{R}^n)$, depending on a parameter $x_s \in E^s(x_0)$:

$$\Gamma_f(\gamma)_k = \begin{cases} (\pi_s(x_0)x_0 + x_s, \pi_u(x_0)f^{-1}(\gamma_1)), & \text{for } k = 0, \\ (\pi_s(x_k)f(\gamma_{k-1}), \pi_u(x_k)f^{-1}(\gamma_{k+1})), & \text{for } k > 0. \end{cases} \quad (1)$$

It is not hard to see that, for x_s small enough, Γ_f is a contraction in a neighbourhood of the orbit x. Let the sequence ν be the unique attracting fixed point of Γ_f near x. Then ν is an orbit in the local stable set of x_0 with $\pi_s(\nu_0) = x_s$. By varying the parameter x_s, we get the local stable set as the graph of a function $E^s(x_0) \to E^u(x_0)$.

Smoothness of the local stable manifold can be proven using the implicit function theorem. The equation to solve for ν_0 is $\pi_u(x_0)[\Gamma_f(\nu)_0 - \nu_0] = 0$. By elementary means one shows that Γ_f as a mapping on $E^s(x_0) \times C_b(\mathbb{N}, \mathbb{R}^n)$ is C^k if $f \in C^k$, see [Irwin,1972], [Shub,1980]. Because Γ_f is a contraction, we can invoke the implicit function theorem to prove smoothness of $W^{s,\epsilon}(x)$.

The local unstable manifold is just the local stable manifold for f^{-1}. ■

For later use in chapter 3, we include the following remark.

Remark 1.2 *The local stable manifold $W^{s,\epsilon}(x)$ depends continuously on x in the C^k topology, if f is C^k. This can be proven with the following trick, which actually provides a variant of the proof of the above theorem, cf. [Shub,1980] for details. Let $C_b(\mathfrak{D}, \mathbb{R}^n)$ be the space of bounded maps $\mathfrak{D} \to \mathbb{R}^n$. Let the map $F : C_b(\mathfrak{D}, \mathbb{R}^n) \to C_b(\mathfrak{D}, \mathbb{R}^n)$ be defined by $F(h) = f \circ h \circ f^{-1}$. Consider F near the inclusion map $i : \mathfrak{D} \hookrightarrow \mathbb{R}^n$, which is a hyperbolic singularity of F. By compactness of $\mathfrak{D}$ one can check that F is as smooth as f. E.g. the stable eigenspace $E^s_F(i)$ is determined by*

$$E^s_F(i) \quad = \quad \{\psi \in C_b(\mathfrak{D}, \mathbb{R}^n) | \psi(x) \in E^s_f(x), \forall x \in \mathfrak{D}\},$$

where again we identify $T_x\mathbb{R}^n \cong \mathbb{R}^n$. Make the local stable manifold $W^s_F(i)$ of F with the construction from the proof of the above theorem. For $x \in \mathfrak{D}$, we have

$$W^s_f(x) \quad = \quad \{\phi(x) | \phi \in W^s_F(i)\}.$$

Because $W^s_F(i)$ is as smooth as f, the claim follows.

For implementation of the construction of local (un)stable manifolds with the contraction (1), described in the proof of the stable manifold theorem above, in a computer, the following remark is of importance.

Remark 1.3 *If we do not know the exact positions of the orbit x_n, we can use the projections π_s, π_u in the points γ_n instead of in x_n, in the expression for Γ_f. The map Γ_f perturbed this way still is a contraction in a neighbourhood of the orbit x.*

Also, it suffices to know the splittings $\mathbb{R}^n = E^s(x) \oplus E^u(x)$ only approximately. A small perturbation of the projections $\pi_s(x), \pi_u(x)$ still gives a contraction in a neighbourhood of an orbit.

We now study the corresponding theory for endomorphisms. Let f be an endomorphism on $\mathbb{R}^n$. A (local) stable set of a point is defined in the same way as for diffeomorphisms. Let $\{x_i\}_{i \in \mathbb{Z}}$, be an orbit, thus $x_{i+1} = f(x_i)$. We define an unstable set of the orbit x as the set of points y, such that the minimal distance between $f^{-i}(y)$ and x_{-i} goes to zero for $i \to \infty$. We speak here of an unstable set of an orbit, instead of an unstable set of a point, since there can be several orbits passing through one point. The unstable sets of two different orbits x, z with $x_0 = z_0$, are different.

As above, let $\mathfrak{D} = f(\mathfrak{D})$ be a compact hyperbolic invariant set. With the contractions described below it is possible to prove that local (un)stable sets in $\mathfrak{D}$ are embedded manifolds, as smooth as f. Notice though, that global (un)stable sets need not be manifolds. Let $\{x_k\}_{k\in\mathbb{Z}}$ be an orbit in $\mathfrak{D}$. We show how to construct the local stable manifold of the point x_0. Recall that $C_b(\mathbb{N}, \mathbb{R}^n)$ denotes the set of bounded sequences $\mathbb{N} \to \mathbb{R}^n$, equipped with the supnorm. Consider the following map Γ_f on $C_b(\mathbb{N}, \mathbb{R}^n)$, depending on a parameter $x_s \in E^s(x_0)$, see [Shub,1980]:

$$\Gamma_f(\gamma)_k =$$

$$\begin{cases} (\pi_s(x_0)x_0 + x_s, \pi_u(x_0)\{\gamma_0 - [Df(x_0)]^{-1}[\gamma_1 - f(\gamma_0)]\}), & \text{for } k = 0, \\ (\pi_s(x_k)f(\gamma_{k-1}), \pi_u(x_k)\{\gamma_k - [Df(x_k)]^{-1}[\gamma_{k+1} - f(\gamma_k)]\}), & \text{for } k > 0. \end{cases}$$

Note that for the expression $[Df(x_k)]^{-1}$ only the unstable coordinates are relevant, i.e. we can view $[Df(x_k)]^{-1}$ as mapping $Df(x_k)E^u(x_k)$ to $E^u(x_k)$. This shows that Γ_f is well defined. Again it is not hard to check that Γ_f is a contraction in a neighbourhood of the orbit x, for x_s small. The first element of the fixed point of Γ_f near x is a point in the local stable manifold and we get all of this local stable manifold by varying x_s.

There is a similar contraction to get orbits: $-\mathbb{N} \to \mathbb{R}^n$ that remain within ϵ-distance of an orbit $x : -\mathbb{N} \to \mathbb{R}^n$ in $\mathfrak{D}$ ($-\mathbb{N}$ is the set of integers which are negative or zero). Let again $x : \mathbb{Z} \to \mathfrak{D}$ be an orbit of f. Consider the map Γ_f on $C_b(-\mathbb{N}, \mathbb{R}^n)$, depending on a parameter $x_u \in E^u(x_0)$:

$$\Gamma_f(\gamma)_k =$$

$$\begin{cases} (\pi_s(x_0)f(\gamma_{-1}), \pi_u(x_0)x_0 + x_u), & \text{for } k = 0, \\ (\pi_s(x_k)f(\gamma_{k-1}), \pi_u(x_k)\{\gamma_k - [Df(x_k)]^{-1}[\gamma_k - f(\gamma_{k+1})]\}), & \text{for } k < 0. \end{cases}$$

Compared with the map used to construct local stable manifolds of points of endomorphisms, we changed the term for $k = 0$ and we consider sequences $-\mathbb{N} \to \mathbb{R}^n$ instead of sequences $\mathbb{N} \to \mathbb{R}^n$. For x_u small, Γ_f is a contraction in a neighbourhood of the orbit x, its fixed point is an orbit $y : -\mathbb{N} \to \mathbb{R}^n$ with y_0 lying in the local unstable set of the orbit x. By varying x_u we get the local unstable manifold of the orbit x.

We can thus construct local stable and unstable manifolds in hyperbolic invariant sets of endomorphisms, yielding the following theorem.

Theorem 1.4 *Let $\mathfrak{D}$ be a compact hyperbolic invariant set for an endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. Then the local stable set $W^{s,\varepsilon}(x)$ of a point $x \in \mathfrak{D}$ and the local unstable set $W^{u,\varepsilon}(x)$ of an orbit x in $\mathfrak{D}$, are embedded disks, as smooth as f.* ■

Finally, we remark that finding the hyperbolic splitting is no problem if we want to construct (local) stable and unstable manifolds of a fixed point, then we project on stable and unstable eigenspaces of the derivative of f. For each computed sequence g, we can check that it really approximates an orbit by comparing γ_k with $f(\gamma_{k-1})$ for all k.

2 Normally hyperbolic invariant manifolds

In this section we discuss some generalizations of the contractions of the previous section to construct other locally invariant manifolds. This section is not needed in the following sections, so the reader may skip this section in first reading.

In the previous section we studied constructions for computing local (un)stable manifolds of points in hyperbolic sets. A local stable manifold consists of points such that the distance between these points goes to zero under iteration. We shall see below how to construct invariant submanifolds of a local stable manifold, such that this distance goes to zero faster then with some prescribed speed. This is called a (local) strong stable manifold [Shub,1980].

We shall restrict to diffeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$. Endomorphsisms can be treated using slightly different contractions, in the same spirit as in the previous section. Let $\mathfrak{D}$ be a compact invariant set, not necessarily hyperbolic. Assume we have, for each $x \in \mathfrak{D}$, a splitting $\mathbb{R}^n = E^1(x) \oplus E^2(x)$. This splitting is assumed to be invariant under Df and to depend continuously on x. We identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$, for all $x \in \mathfrak{D}$. Suppose there is a positive constant λ, such that $\frac{1}{\lambda}Df(x)|_{E^1(x)}$ is a contraction and such that $\frac{1}{\lambda}Df(x)|_{E^2(x)}$ is an expansion. Because we only want to explain the construction of a local strong stable manifold, we assume $\lambda < 1$. Denote by $\pi_1(x), \pi_2(x)$ the canonical projections on $E^1(x), E^2(x)$. We write $y = (\pi_1(x)y, \pi_2(x)y)$ (as before, the projections are centred in 0, not in x). Below we discuss the construction of a local invariant manifold $W^1(x)$, with tangent space $T_xW^1(x) = E^1(x)$. See also [Irwin,1972].

Let $C(\mathbb{N}, \mathbb{R}^n)$ denote the set of all, so possibly unbounded, sequences $\mathbb{N} \to \mathbb{R}^n$. For $x \in C(\mathbb{N}, \mathbb{R}^n)$, we make use of the map $A_x : C(\mathbb{N}, \mathbb{R}^n) \to C(\mathbb{N}, \mathbb{R}^n)$, defined by

$$A_x(\gamma)_k = x_k - \lambda^k(\gamma_k - x_k),$$

A_x is thus a scaling around the orbit x.

Let $\Gamma_f : C(\mathbb{N}, \mathbb{R}^n) \to C(\mathbb{N}, \mathbb{R}^n)$, depending on a parameter $x_1 \in E^1(x_0)$, be defined as in section 1:

$$\Gamma_f(\gamma)_k = \begin{cases} (\pi_1(x_0)x_0 + x_1, \pi_2(x_0)f^{-1}(\gamma_1)), & \text{for } k = 0, \\ (\pi_1(x_k)f(\gamma_{k-1}), \pi_2(x_k)f^{-1}(\gamma_{k+1})), & \text{for } k > 0. \end{cases}$$

Finally, consider the map $\Upsilon_f : C_b(\mathbb{N}, \mathbb{R}^n) \to C_b(\mathbb{N}, \mathbb{R}^n)$:

$$\Upsilon_f(\gamma) = A_x^{-1} \circ \Gamma_f \circ A_x.$$

One easily checks that Υ_f is well defined and is a contraction in a neighbourhood of the orbit x for x_1 small. If $\nu \in C_b(\mathbb{N}, \mathbb{R}^n)$ is the fixed point of Υ_f near

x, then $A_x(\nu)$ is an orbit of f. By varying the parameter x_1, we get the unique local invariant manifold $W^1(x_0)$. This manifold is as smooth as f. We leave a discussion of robustness with respect to numerical errors to the reader.

With a similar contraction we can get local strong unstable manifolds.

3 Invariant sets

Let, as in the first section, $\mathfrak{D}$ be a compact hyperbolic invariant set of an endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. The contractions described in section 1 can be adjusted to construct orbits in $\mathfrak{D}$. We show this for diffeomorphisms f.

Consider the following map Γ_f on $C_b(\mathbb{Z}, \mathbb{R}^n)$:

$$(\Gamma_f(\gamma))_k = (\pi_s(\gamma_k)f(\gamma_{k-1}), \pi_u(\gamma_k)f^{-1}(\gamma_{k+1})). \tag{2}$$

This map is a contraction in a neighbourhood of orbits in $\mathfrak{D}$, these orbits are fixed points of $\mathfrak{D}$. We discuss this construction in more detail in the next section for a diffeomorphism possessing a horseshoe. Section 5 then contains a numerical example, where we apply this construction to find orbits in a horseshoe for the Henon map.

The remainder of this section contains some additional material, not directly concerning the main topic of this paper. Above and in section 1, we assumed the existence of a continuous splitting in stable and unstable directions and defined contractions like (1) and (2) to construct orbits in the hyperbolic set and their stable and unstable manifolds. Now we ask what information can be derived from knowledge that a map Γ_f as in (2), for some projections π_s, π_u, is a contraction. Compare also [HP,1970], [Moser,1973] and [PT,1993].

Suppose given splittings $\mathbb{R}^n = E_s(x) \oplus E_u(x)$, depending continuously on x, for $x \in \mathbb{R}^n$. Let $\pi_s(x) : \mathbb{R}^n \to E_s(x)$ and $\pi_u(x) : \mathbb{R}^n \to E_u(x)$ denote the corresponding projections, where e.g. the kernel of $\pi_s(x)$ equals $E_u(x)$. Let $\Gamma_f : C_b(\mathbb{Z}, \mathbb{R}^n) \to C_b(\mathbb{Z}, \mathbb{R}^n)$ be defined as in (2).

Theorem 3.1 *The maximal invariant set $\mathfrak{D}$ of f in a compact subset $S \subset \mathbb{R}^n$ is hyperbolic, if Γ_f is a contraction, locally near orbits in $\mathfrak{D}$.*

PROOF. The maximal invariant set $\mathfrak{D}$ in S and its stable and unstable manifolds can be constructed with the contractions (1), (2) described earlier. The tangent spaces of the stable and unstable manifolds provide the splitting in stable and unstable directions. The only thing which remains to be shown is the continuity of this splitting.

Recall from remark 1.2, that $C_b(\mathfrak{D}, \mathbb{R}^n)$ denotes the space of bounded maps $\mathfrak{D} \to \mathbb{R}^n$ and that $F : C_b(\mathfrak{D}, \mathbb{R}^n) \to C_b(\mathfrak{D}, \mathbb{R}^n)$ is the map $F(h) = f \circ h \circ f^{-1}$. The inclusion $i : \mathfrak{D} \hookrightarrow \mathbb{R}^n$ is a fixed point of F. Write $C_b(\mathfrak{D}, \mathbb{R}^n) = \mathcal{E}_s \times \mathcal{E}_u$, where

$$\mathcal{E}_s = \{\psi \in C_b(\mathfrak{D}, \mathbb{R}^n) | \pi_u(q)\psi(q) = 0, \forall q \in \mathfrak{D}\}$$

and $\mathcal{E}_u$ is defined similarly with π_u replaced by π_s. Denote the corresponding projections to $\mathcal{E}_s$ and $\mathcal{E}_u$, with kernel $\mathcal{E}_u$ resp. $\mathcal{E}_s$, by Π_s and Π_u, so e.g.

$$\Pi_s(\pi_s(q)\psi(q), \pi_u(q)\psi(q)) = (\pi_s(q)\psi(q), 0).$$

Let $\mathcal{C}$ be the space of bounded sequences $\mathbb{Z} \to C_b(\mathfrak{D}, \mathbb{R}^n)$. Define $\Gamma_F : \mathcal{C} \to \mathcal{C}$ by

$$\Gamma_F(\chi)_k = (\Pi_s F(\chi_{k-1}), \Pi_u F^{-1}(\chi_{k+1})).$$

It is easily recognised that, since Γ_f is a contraction, also Γ_F is a contraction.

The fixed point i of F is thus a hyperbolic one and also, since F is smooth, the local stable and unstable manifolds of F near i are smooth, cf. [Shub,1980]. It follows that the splitting in stable and unstable directions along $\mathfrak{D}$ is continuous, compare remark 1.2. ■

4 The Smale horseshoe

To illustrate the theory of section 1, we shall work things out for a diffeomorphism possessing a horseshoe [6]. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism, such that a square S is mapped over itself in a horseshoe shape as indicated in figure 1 below. Within S, f is (almost) linear, expanding in the vertical direction and contracting in the horizontal direction. We have indicated vertical and horizontal rectangles which satisfy $f(H_0) = V_0, f(H_1) = V_1, f^{-1}(V_0) = H_0, f^{-1}(V_0) = H_0$.

The square S contains a maximal invariant set whose dynamics can be described using symbolic dynamics. There is a 1-1 correspondence between sequences $\mathbb{Z} \to \{0, 1\}$ and orbits in this invariant set, if we associate to an orbit $\{f^k(p)\}$, the sequence $\{\mathfrak{S}_k(p)\}_{k\in\mathbb{Z}}$ defined by

$$\mathfrak{S}_k(p) = j \text{ if } f^k(p) \in V_j.$$

The maximal invariant set in S is not an attracting set; most points in S leave this box after some iterations, since the invariant set has zero measure (assuming f is C^2). Let us explain how we can approximate orbits in this invariant set. Denote by π_x, π_y the projection to the x-axis resp. the y-axis. Define the map $\Gamma_f : C_b(\mathbb{Z}, S) \to C_b(\mathbb{Z}, S)$ by

$$\Gamma_f(\gamma)_k = (\pi_x f(\gamma_{k-1}), \pi_y f^{-1}(\gamma_{k+1})).$$

Theorem 4.1 *The map $\Gamma_f : C_b(\mathbb{Z}, S) \to C_b(\mathbb{Z}, S)$ is a contraction, locally near orbits in the maximal invariant set $\mathfrak{D}$ of f in S. So each orbit in $\mathfrak{D}$ is an attracting fixed point of Γ_f.*

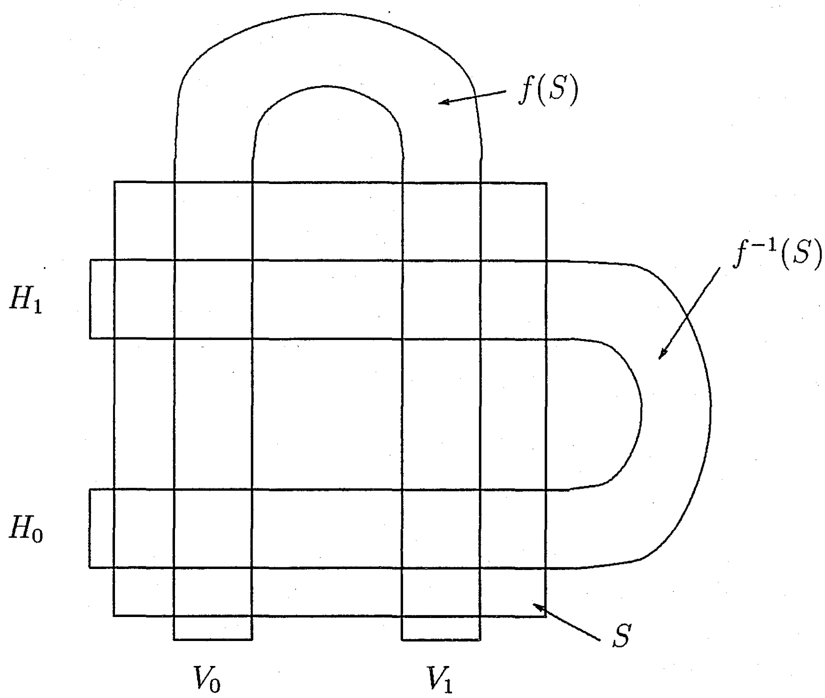

Figure 1: The Smale horsehoe.

PROOF. The maximal invariant set $\mathfrak{D}$ is contained in T, where

$$T = \bigcup_{i,j=0,1} V_i \cap H_j.$$

Since both $f(T)$ and $f^{-1}(T)$ lie in S, it is easily recognised that Γ_f is a contraction, locally near an orbit in $\mathfrak{D}$. In fact, Γ_f contracts the distance between each two sequences that have their i^{th} point in the same component of T. Given a symbolic sequence $\{\mathfrak{S}_n\}_{n\in\mathbb{Z}}$, choose a sequence $\gamma \in C_b(\mathbb{Z}, S)$ according to the principle

$$\gamma_k \in V_i \cap H_j, \text{ where } i = \mathfrak{S}_k, j = \mathfrak{S}_{k+1}.$$

Under iteration of Γ_f, this sequence converges to the unique orbit with the given symbolic coding. The reason for this is simply that $\Gamma_f(\gamma)_k$ is in the same component of T as γ_k, by the above restriction on γ. ∎

Consider now the problem of approximating orbits in a horseshoe with a computer. It follows from theorem 4.1 and its proof that we can take a symbolic sequence, choose an appropriate starting sequence in $C_b(\mathbb{Z}, T)$ and then compute the orbit corresponding to this symbolic sequence by iterating the starting sequence under Γ_f. On a computer we can only work with finite sequences and finite precision. By working with periodic sequences (then we have only a finite number of points to deal with), we can approximate periodic orbits up to any desired precision. We can also approximate long pieces of nonperiodic orbits with some prescribed symbolic coding up to a desired precision: Let $\{\mathfrak{S}_k\}$ be a symbolic sequence and consider the set $P = \{x \in \mathbb{R}^n | f^k(x) \in V_k \text{ for } -N < k < N\}$. It is easily seen that P is contained in a cube of width 2^{-N} . This makes it possible to approximate long pieces of orbits with some prescribed symbolic coding by approximating a periodic orbit with large period.

It is further possible to compute points whose past or future is contained in $[0,1]$. These are points lying on unstable resp. stable manifolds of points in the invariant set. To get a point whose future stays in $[0,1]$, we work with sequences $\mathbb{N} \to T$ and we define a map Γ_f on $C_b(\mathbb{N}, T)$

$$\Gamma_f(\gamma)_k = \begin{cases} (x, \pi_y f^{-1}(\gamma_1)), & \text{for } k = 0, \\ (\pi_x f(\gamma_{k-1}), \pi_y f^{-1}(\gamma_{k+1})), & \text{for } k > 0. \end{cases}$$

This is the same map as before, except that we now have some kind of initial condition. The parameter x is the stable coordinate of the first point of the orbit we want to construct. Note that we can approximate local stable manifolds by varying the parameter x. Again, since a computer can only deal with a finite number of points, we can compute local stable manifolds of periodic points with a computer (with some prescribed accuracy). Local unstable manifolds can be constructed in a similar way.

To end this section, we apply the results from section 3. Suppose f has a topological horseshoe. That is, f maps a square S twice over itself such that $f(S) \cap \partial S$ consists of four intervals in the horizontal boundaries of S, as drawn in figure 1. There is a surjection between the invariant set of f in S and the set of sequences of symbols $\{0,1\}^{\mathbb{Z}}$. For applications it is important to be able to decide whether this surjection is actually a bijection and whether the invariant set is hyperbolic. Combining the mean value theorem with the results in the previous section, we have the following, compare [Moser,1973], [PT,1993].

Remark 4.2 *Suppose there is a constant $\lambda < 1$ with*

$$\begin{aligned} \forall x \in S \cap f^{-1}(S), \quad & \|\pi_x Df(x)v\| \le \lambda \|v\|, \\ \forall x \in S \cap f(S), \quad & \|\pi_y Df^{-1}(x)v\| \le \lambda \|v\|. \end{aligned}$$

Then the maximal invariant set of f in S is a hyperbolic horseshoe.

More geometrically, the above conditions can be described as the existence of bundles of invariant stable and unstable cones, see [PT,1993].

5 Numerical examples

We performed some computations using the methods of this paper. Figure 2 below illustrates a computation done for the Henon map, which in suitable coordinates takes the form $h(x,y) = (a - by - x^2, x)$. For the projection $\pi_x(q)$, described in the previous sections, we used the projection on the stable eigendirection of the linear map $x \to Dh(h^{-1}(q))x$. Similarly for $\pi_y(q)$ we used the projection on the unstable eigendirection of this linear map. We computed a periodic orbit of the Henon map with parameters $a = 3, b = 0.3$, for which there is a horseshoe. We have computed a periodic orbit with period 300, to get an idea of the position of the horseshoe. We checked that the computed points really approximate this periodic orbit by comparing each point with the image under h of the previous point.

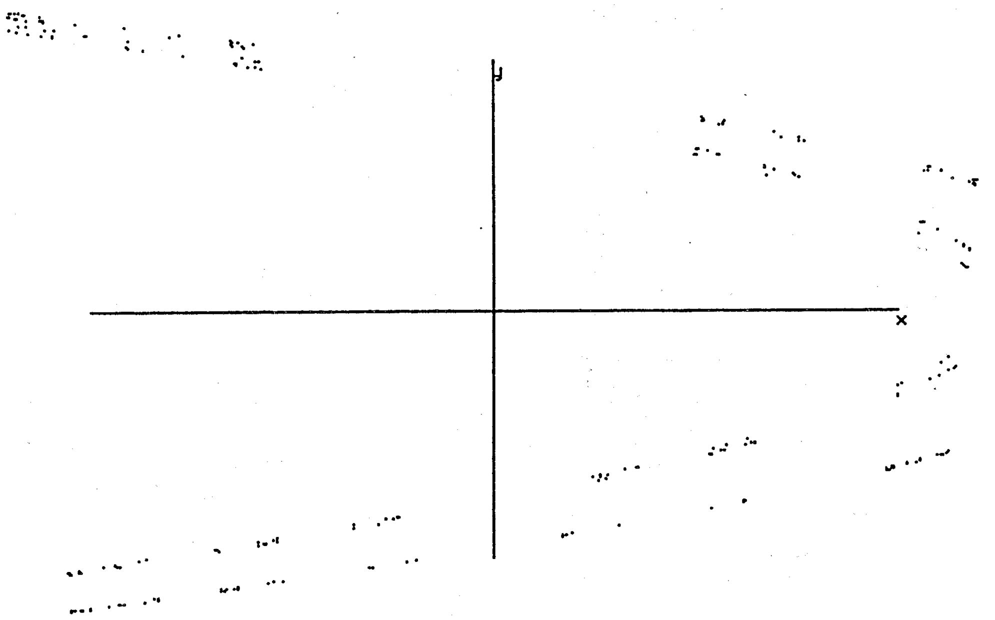

Figure 2: The horseshoe in the Henon map $h(x,y) = (a - by - x^2, x), a = 3, b = 0.3$.

Figure 3 illustrates an unstable manifold of a three dimensional endomorphism. We first computed several points on the local unstable manifold. The local unstable manifold consists of points for which the unstable x-coordinate lies between -1 and 1. We then iterated these points seven times.

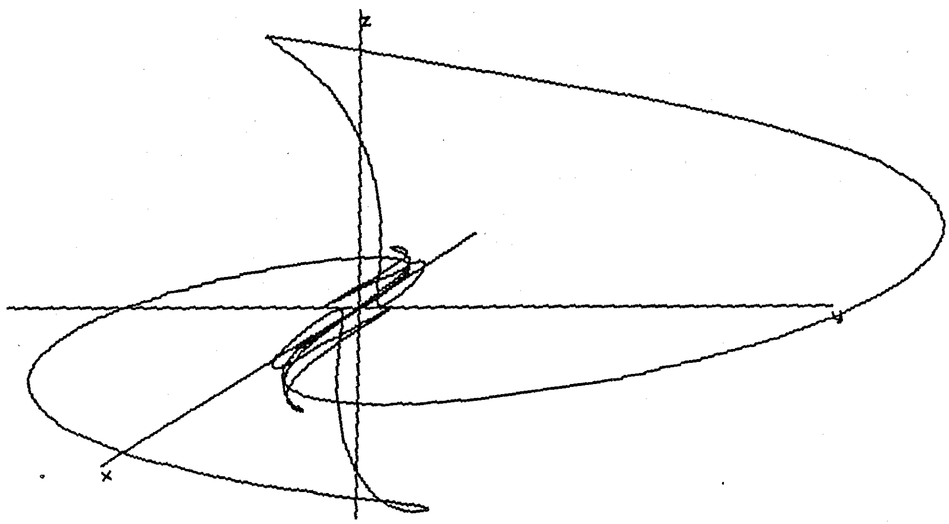

Figure 3: Unstable manifold of the map $(x,y,z) \to (2x - \frac{1}{5}(x + \frac{1}{3}y + \frac{1}{24}z)^3, -\frac{1}{5}(y - \frac{1}{4}z) + \frac{3}{10}x^3, \frac{1}{5}(4y+z))$.

References

[HP,1970] M. Hirsch, C. Pugh, Stable manifolds and hyperbolic sets, *Proc. Symp. Pure Math.* **14** (1970), 133-163.

[HOV,1993] A.J. Homburg, H.M. Osinga, G. Vegter, Numerical computation of invariant manifolds, preprint.

[Irwin,1970] M.C. Irwin, On the stable manifold theorem, *Bull. London Math. Soc.* **2** (1970), 196-199.

[Irwin,1972] M.C. Irwin, On the smoothness of the composition map, *Quart. J. Math. Oxford* **23** (1972), 113-133.

[Irwin,1980] M.C. Irwin, *Smooth dynamical systems*, Academic press, London, (1980).

[Moser,1973] J. Moser, *Stable and random motions in dynamical systems*, Princeton Univ. Press, Princeton, (1973).

[NY,1988] H.E.Nusse, J.A. Yorke, A procedure for finding numerical trajectories on chaotic saddles, Physica D **36** (1989), 137-156.

[Perron,1929] O. Perron, Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen, *J. Reine Angew. Math.* **161** (1929), 41-64.

[PT,1993] J. Palis, F. Takens, *Homoclinic bifurcations: hyperbolicity, fractional dimensions and infinitely many attractors*, Cambridge Univ. Press (1993).

[Shub,1980] M. Shub, *Global stability of dynamical systems*, Springer Verlag, Berlin (1980).

[Smale,1967] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747-817.

[YKJ,1991] Z. You, E.J. Kostelich, J.A. Yorke, Calculating stable and unstable manifolds, *Bifurcation and chaos* **1** no. 3 (1991), 605-623.

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